

SOME SOLUTIONS FOR ASSIGNMENT 7

8.19. Suppose that $f : \mathbb{R} \rightarrow \mathbb{C}$ is continuous and 2π -periodic and that $\alpha \in \mathbb{R}$ with α/π irrational, and define

$$A_{N,x}(f) = \frac{1}{N} \sum_{n=1}^N f(x + n\alpha)$$

and

$$c_0(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) dt$$

($c_0(f)$ is the usual Fourier coefficient). We wish to prove that for any $x \in \mathbb{R}$,

$$\lim_{N \rightarrow \infty} A_{N,x}(f) = c_0(f). \quad (7.1)$$

We first take $f(t) = e^{ikt}$. If $k = 0$ then $f(t) = 1$ for all t and $A_{N,x}(f) = c_0(f) = 1$, verifying (7.1). If $k \neq 0$ then

$$c_0(f) = \frac{e^{ikt}}{2\pi ik} \Big|_{-\pi}^{\pi} = 0,$$

and since α/π is irrational and hence $e^{ik\alpha} \neq 1$ for $k \neq 0$,

$$\lim_{N \rightarrow \infty} A_{N,x}(f) = \lim_{N \rightarrow \infty} \frac{e^{ikx} e^{i(N+1)k\alpha} - e^{ik\alpha}}{N(e^{ik\alpha} - 1)} = 0.$$

Thus (7.1) holds when $f(x) = e^{ikx}$. Since both sides are linear in f it also holds if $f = P$, where P is a trigonometric polynomial of the form $P(t) = \sum_{k=-M}^M a_k e^{ikt}$, that is, given P and $\epsilon > 0$ there is an integer $N_P = N_P(\epsilon)$ such that $|A_{N,x}(P) - c_0(P)| < \epsilon$ if $N \geq N_P$.

Now suppose that f is an arbitrary continuous, 2π -periodic function. We know by the Stone-Weierstrass Theorem (see Theorem 8.15) that for any $\epsilon > 0$ there is a trigonometric polynomial P such that $|f(t) - P(t)| < \epsilon$ for all t . From this we observe that

$$|c_0(f) - c_0(P)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t) - P(t)| dt < \epsilon$$

and

$$|A_{N,x}(f) - A_{N,x}(P)| \leq \frac{1}{N} \sum_{n=1}^N |f(x + n\alpha) - P(x + n\alpha)| < \epsilon.$$

Then for $N \geq N_P$,

$$|A_{N,x}(f) - c_0(f)| \leq |A_{N,x}(f) - A_{N,x}(P)| + |A_{N,x}(P) - c_0(P)| + |c_0(P) - c_0(f)| < 3\epsilon,$$

so that (7.1) holds. ■