SOME SOLUTIONS FOR ASSIGNMENT 8

9.14 We are studying

$$f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2}, & \text{if } (x,y)\neq(0,0),\\ 0, & \text{if } (x,y)=(0,0).\end{cases}$$

(b) Let $\mathbf{u}=(v,w)$ be a unit vector in $\mathbb{R}_2$, so that $v^2+w^2=1$. Then the directional derivative of f at $(0,0)=\mathbf{0}$ in the direction $\mathbf{u}$ is

$$D_{\mathbf{u}}f(\mathbf{0})=\lim_{s\to0}\frac{f(\mathbf{0}+s\mathbf{u})-f(\mathbf{0})}{s}=\lim_{s\to0}\frac{f(sv,sw)}{s}=\lim_{s\to0}\frac{(sv)^3}{s((sv)^2+sw)^2}=\frac{v^3}{v^2+w^2}=v^3.$$

Clearly $|D_{\mathbf{u}}f(\mathbf{0})|\le|v|\le|\mathbf{u}|=1$.

(a) The partial derivative D_1f is just $D_{(1,0)}f$, and similarly $D_2f=D_{(0,1)}f$. So from (b), $D_1f(0,0)=1$ and $D_2f(0,0)=0$. At a point (x,y) other than $(0,0)$ we may use the elementary rules of differentiation, which lead to:

$$D_1f(x,y)=\frac{x^2(x^2+3y^2)}{(x^2+y^2)^2},\qquad D_1f(x,y)=-\frac{2x^3y}{(x^2+y^2)^2}.\tag{8.1}$$

Then for $(x,y)\neq(0,0)$, $|D_1f(x,y)|\le3$ because $x^2(x^2+3y^2)\le(x^2+y^2)(3x^2+3y^2)$, and $|D_2f(x,y)|\le1$ because $|x^3y|\le x^2(x^2+y^2)/2\le(x^2+y^2)^2/2$; moreover, $D_1f(0,0)$ and $D_2f(0,0)$ satisfy these same bounds. Here we have used the inequality $|xy|\le(x^2+y^2)/2$, which follows from $(x\pm y)^2\ge0$.

Remark 1: Since from (8.1) it is clear that $D_1f(x,y)$ and $D_2f(x,y)$ are continuous for $(x,y)\neq(0,0)$, it follows from Theorem 9.21 that f is differentiable at (x,y) if $(x,y)\neq(0,0)$.

(c) We consider $\boldsymbol{\gamma}:\mathbb{R}^1\to\mathbb{R}^2$ a differentiable function with $\boldsymbol{\gamma}(0)=\mathbf{0}$, and set $g(t)=f(\boldsymbol{\gamma}(t))$. We will prove that g is differentiable and that if $\boldsymbol{\gamma}'$ is continuous at some point t_0 then so is g', from which it follows that if $\boldsymbol{\gamma}$ is C^1, then so is g.

Remark 2: Rudin assumes that $|\boldsymbol{\gamma}'(0)|\neq0$, but I do not know why he does so. For, as we will see, the result is true without this assumption. On the other hand, the assumption does not make the problem any easier because, although the assumption simplifies the proof that $\boldsymbol{\gamma}'(0)$ exists, we are to prove that $\boldsymbol{\gamma}'(t)$ exists for all t, and it may happen that for some $t_0\neq0$, $\boldsymbol{\gamma}(t_0)=0$ and $\boldsymbol{\gamma}'(t_0)=\mathbf{0}$, so that whatever problem was avoided at $t=0$ resurfaces at $t=t_0$.

Proof: Whenever t is such that $\boldsymbol{\gamma}(t)\neq(0,0)$, existence of $g'(t)$ follows from the Remark 1 and the chain rule, with

$$\begin{aligned}g'(t)&=D_1f(\boldsymbol{\gamma}(t))\gamma_1'(t)+D_2f(\boldsymbol{\gamma}(t))\gamma_2'(t)\\&=\frac{\gamma_1(t)^2(\gamma_1(t)^2+3\gamma_2(t)^2)}{(\gamma_1(t)^2+\gamma_2(t)^2)^2}\gamma_1'(t)-\frac{2\gamma_1(t)^3\gamma_2(t)}{(\gamma_1(t)^2+\gamma_2(t)^2)^2}\gamma_2'(t).\end{aligned}\tag{8.2}$$

Note that if $\boldsymbol{\gamma}(t_0) \neq 0$ then (8.2) holds for in a neighborhood of t_0, and that therefore if $\boldsymbol{\gamma}'$ is continuous at t_0 then so is g'.

Now we fix some $t_0 \in \mathbb{R}$ with $\boldsymbol{\gamma}(t_0) = \mathbf{0}$ (for example, possibly $t_0 = 0$) and show that $g'(t_0)$ exists and that if $\boldsymbol{\gamma}'$ is continuous at t_0 then so is g'. We distinguish two cases:

Case 1: $\boldsymbol{\gamma}'(t_0) \neq \mathbf{0}$. (This is the case one would consider for the problem as Rudin states it, taking $t_0 = 0$, if one did not notice the difficulty raised in Remark 2 above.) We calculate $g'(t_0)$ directly, using the fact that $\boldsymbol{\gamma}'(t_0) \neq \mathbf{0}$ implies $\boldsymbol{\gamma}(t_0 + h) \neq \mathbf{0}$ for h nonzero and sufficiently small:

$$\begin{aligned} g'(t_0) = \lim_{h\to 0} \frac{g(t_0+h)}{h} &= \lim_{h\to 0} \frac{\gamma_1(t_0+h)^3}{h(\gamma_1(t_0+h)^2 + \gamma_2(t_0+h)^2)} \\ &= \lim_{h\to 0} \frac{(\gamma_1(t_0+h)/h)^3}{(\gamma_1(t_0+h)/h)^2 + \gamma_2(t_0+h)/h)^2} \\ &= \frac{\gamma_1'(t_0)^3}{\gamma_1'(t_0)^2 + \gamma_2'(t_0)^2}. \end{aligned} \tag{8.3}$$

If $\boldsymbol{\gamma}'$ is continuous at t_0 then, since for t near t_0 but $t \neq t_0$, $g'(t)$ is given by (8.2),

$$\lim_{t\to t_0} g'(t) = \lim_{t\to t_0} \left[\frac{\gamma_1(t)^2(\gamma_1(t)^2 + 3\gamma_2(t)^2)}{(\gamma_1(t)^2 + \gamma_2(t)^2)^2}\gamma_1'(t) - \frac{2\gamma_1(t)^3\gamma_2(t)}{(\gamma_1(t)^2+\gamma_2(t)^2)^2}\gamma_2'(t) \right]. \tag{8.4}$$

In each term on the right side of (8.4) we divide numerator and denominator by $(t - t_0)^4$, and use the fact that $\gamma_i'(t_0) = \lim_{t\to t_0} \gamma_i(t)/(t - t_0)$ for $i = 1, 2$; with (8.3) this yields

$$\lim_{t\to t_0} g'(t) = \frac{\gamma_1'(t_0)^3(\gamma_1'(t_0)^2 + 3\gamma_2'(t_0)^2)}{(\gamma_1'(t_0)^2 + \gamma_2'(t_0))^2)^2} - \frac{2\gamma_1'(t_0)^3\gamma_2'(t_0)^2}{(\gamma_1'(t_0)^2+\gamma_2'(t_0)^2)^2} = g'(t_0).$$

Case 2: $\boldsymbol{\gamma}'(t_0) = \mathbf{0}$. Consider the difference quotient $[g(t_0+h) - g(t_0)]/h = f(\boldsymbol{\gamma}(t_0+h))/h$. If $\gamma_1(t_0 + h) \neq 0$ then

$$\left|\frac{f(\boldsymbol{\gamma}(t_0+h))}{h}\right| = \left|\frac{\gamma_1(t_0+h)^3}{h(\gamma_1(t_0+h)^2 + \gamma_2(t_0+h)^2)}\right| \leq \left|\frac{\gamma_1(t_0+h)^3}{h\gamma_1(t_0+h)^2}\right| = \left|\frac{\gamma_1(t_0+h)}{h}\right|; \tag{8.5}$$

if $\gamma_1(t_0+h) = 0$ then $f(\boldsymbol{\gamma}(t_0+h)) = 0$ and (8.5) holds trivially. Since $\lim_{h\to 0} \gamma_1(t_0+h)/h = \gamma_1'(t_0) = 0$,

$$g'(t_0) = \lim_{h\to 0} \frac{f(\boldsymbol{\gamma}(t_0+h))}{h} = 0. \tag{8.6}$$

Finally we must show that if $\lim_{t\to t_0} \boldsymbol{\gamma}'(t) = \mathbf{0}$ then $\lim_{t\to t_0} g'(t) = 0$. But if t is such that (8.2) holds then from the bounds on $D_1 f$ and $D_2 f$ obtained in (a),

$$|g'(t)| \leq 3|\gamma_1(t)| + |\gamma_2(t)| \leq 4|\boldsymbol{\gamma}'(t)|;$$

if (8.3) applies then $|g'(t)| \leq |\gamma_1(t)| \leq |\boldsymbol{\gamma}'(t)|$; and under (8.6), $g'(t) = 0$. Thus always $|g'(t)| \leq 4|\boldsymbol{\gamma}'(t)|$ and the result follows. ■