

### Derivation of Stirling's formula

In these notes we give full details of the derivation of Stirling's formula which was discussed partially in class. We begin by writing

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty t^x e^{-t} dt \\ &= x^{x+1} \int_0^\infty s^x e^{-xs} ds \\ &= x^{x+1/2} e^{-x} \int_{-\sqrt{x}}^\infty \exp \left\{ x \left[ \log \left( 1 + \frac{u}{\sqrt{x}} \right) - \frac{u}{\sqrt{x}} \right] \right\} du,\end{aligned}\tag{1}$$

where we have introduced first  $s = xt$  and then  $u = \sqrt{x}(s - 1)$ . If we define

$$\alpha_x(u) = x \left[ \log \left( 1 + \frac{u}{\sqrt{x}} \right) - \frac{u}{\sqrt{x}} \right],$$

and

$$\psi_x(u) = \begin{cases} \exp \alpha_x(u), & \text{if } u > -\sqrt{x}, \\ 0, & \text{if } u \leq -\sqrt{x}, \end{cases}$$

then (1) becomes

$$\frac{\Gamma(x+1)}{x^{x+1/2} e^{-x}} = \int_{-\infty}^\infty \psi_x(u) du.$$

Now we claim that

- (a)  $\lim_{x \rightarrow \infty} \psi_x(u) = e^{-u^2/2}$ , with uniform convergence on any closed interval  $[-a, a]$ , and
- (b) there exists a positive function  $g(u)$  defined for  $u \in \mathbb{R}$  such that (i)  $\psi_x(u) \leq g(u)$  for all  $u \in \mathbb{R}$  whenever  $x \geq 1$ , and (ii)  $\int_{-\infty}^\infty g(u) du < \infty$ .

Once this claim has been established, we have from Exercise 7.12 of Rudin that Stirling's formula holds in the form

$$\lim_{x \rightarrow \infty} \frac{\Gamma(x+1)}{x^{x+1/2} e^{-x}} = \int_{-\infty}^\infty e^{-u^2/2} du = \sqrt{2\pi}.\tag{2}$$

More precisely, that exercise implies that for any sequence  $(x_n)_{n \in \mathbb{N}}$  of positive numbers with  $\lim_{n \rightarrow \infty} x_n = \infty$ ,

$$\lim_{n \rightarrow \infty} \frac{\Gamma(x_n+1)}{x_n^{x_n+1/2} e^{-x_n}} = \int_{-\infty}^\infty e^{-u^2/2} du,$$

and this in turn implies (2).

To verify part (a) of the claim, notice that for  $x > a^2$ ,  $\psi_x$  is defined as  $\psi_x(u) = \exp \alpha_x(u)$  throughout the interval  $[-a, a]$ , and hence it suffices to show that  $\lim_{x \rightarrow \infty} \alpha_x(u) = -u^2/2$

uniformly on  $[a, a]$ . Now by Taylor's theorem (Theorem 5.15), if  $v > -1$  then there is a number  $v_*$  between 0 and  $v$  such that

$$\log(1+v) = v - \frac{v^2}{2} + \frac{v^3}{3(1+v_*)^3}, \quad (3)$$

and so if  $|v| < 1/2$  then

$$\left| \log(1+v) - \left( v - \frac{v^2}{2} \right) \right| < \frac{8}{3} |v|^3.$$

Thus if  $\sqrt{x} > a/2$  and  $u \in [-a, a]$ ,

$$\left| \alpha_x(u) - \left( -\frac{u^2}{2} \right) \right| < \frac{8a^3}{3\sqrt{x}},$$

which gives the desired uniform convergence.

To check (b) we need to consider separately the cases  $u \leq 0$  and  $u > 0$ . When  $-\sqrt{x} < u \leq 0$  we have by (3) that  $\alpha_x(u) \leq -u^2/2$ , so that whenever  $u \leq 0$ ,

$$0 \leq \psi_x(u) \leq e^{-u^2/2}.$$

This proves (b) for  $u \leq 0$ , with  $g(u) = \exp(-u^2/2)$ .

For the case  $u > 0$  we first fix  $u$  and show that then  $\alpha_x(u)$ , and therefore also  $\psi_x(u)$ , are *decreasing* functions of  $x$ . By a straightforward calculation,

$$\frac{d}{dx} \alpha_x(u) = \log \left( 1 + \frac{u}{\sqrt{x}} \right) - \frac{u}{2(\sqrt{x}+u)} - \frac{u}{2\sqrt{x}}, \quad \text{and} \quad \frac{d^2}{dx^2} \alpha_x(u) = \frac{u^3}{4x^{3/2}(\sqrt{x}+u)}.$$

Now since  $u > 0$ ,  $d^2 \alpha_x(u)/dx^2 > 0$ , so that  $d\alpha_x(u)/dx$  is increasing in  $x$  for all  $u > 0$ . But an application of Taylor's theorem as in (3) (but carried to one more order) yields a bound of the form  $d\alpha_x(u)/dx \leq -v^3/6 + Kv^4$  valid for  $v \leq 1/2$ , where  $v = u/\sqrt{x}$  and  $K$  is some easily computable constant, so that  $d\alpha_x(u)/dx < 0$  for  $x$  large and hence for all  $x$ . From this we have that for  $x \geq 1$ ,  $0 \leq \psi_x(u) \leq g(u)$  with  $g(u) = \psi_1(u)$ . But

$$\alpha_1(u) = \log(1-u) - u \leq A - \frac{u}{2},$$

where  $A = \sup_{u \geq 0} [\log u - u/2]$ ; note that  $A < \infty$  because  $\log(1+u) < u/2$  for  $u$  large. Thus  $g(u) = \psi_1(u) \leq e^A e^{-u/2}$  and so  $\int_0^\infty g(u) du < \infty$ . ■