Practice problems for the Final, part 3

Note: Practice problems for the Final Exam, part 1 and part 2 are the same as Practice problems for Midterm 1 and Midterm 2.

1. Calculate Fourier Series for the function $f(x)$, defined on $[-2, 2]$, where

$$f(x) = \begin{cases} -1, & -2 \leq x \leq 0, \\ 2, & 0 < x \leq 2. \end{cases}$$

2. Calculate Fourier Series for the function $f(x)$, defined on $[-5, 5]$, where

$$f(x) = 3H(x-2).$$

3. Calculate Fourier Series for the function, $f(x)$, defined as follows:

   (a) $x \in [-4, 4]$, and

$$f(x) = 5.$$

   (b) $x \in [-\pi, \pi]$, and

$$f(x) = 21 + 2\sin 5x + 8\cos 2x.$$

   (c) $x \in [-\pi, \pi]$, and

$$f(x) = \sum_{n=1}^{8} c_n \sin nx, \quad \text{with } c_n = 1/n.$$

   (d) $x \in [-3, 3]$, and

$$f(x) = -4 + \sum_{n=1}^{6} c_n(\sin(\pi n x/3) + 7\cos(\pi n x/3)), \quad \text{with } c_n = (-1)^n.$$

4. (a) Let $f(x) = x + x^3$ for $x \in [0, \pi]$. What coefficients of the Fourier Series of f are zero? Which ones are non-zero? Why?

   (b) Let $g(x) = \cos(x^5) + \sin(x^2)$. What coefficients of the Fourier Series of g are zero? Which ones are non-zero? Why?

5. Let $f(x) = 2x + x^4$ for $x \in [0, 5]$.

   (a) Write down the function $G(x)$, which is the odd continuation for $f(x)$. Specify what terms will be zero and non-zero in the Fourier expansion for $G(x)$.

   (b) Write down the function $V(x)$, which is the even continuation for $f(x)$. Specify what terms will be zero and non-zero in the Fourier expansion for $V(x)$.

6. Suppose $f(x)$ is defined for $x \in [0, 7]$, and $f(x) = 2e^{-4x}$. Another function, $F(x)$, is given by the following:

$$F(x) = \sum_{n=0}^{\infty} a_n \cos(\pi n x/7),$$

   where

$$a_n = \frac{2}{7}\int_0^7 2e^{-4x} \cos\left(\frac{\pi n x}{7}\right) dx.$$

   What is the value of $F(3)$? What is the value of $F(-2)$?

7. Let us supposed that both ends of a string of length $25\,cm$ are attached to fixed points at hight 0. Initially, the string is at rest, and has the shape $4\sin(2\pi x/25)$, where x is the horizontal coordinate along the string, with zero at the left end. The speed of wave propagation along the string is $3\,cm/sec$. Write down the complete initial and boundary value problem for the shape of the string.

8. Let us suppose that the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = 50\frac{\partial^2 y}{\partial x^2}, \quad x \in [0, 100], \tag{1}$$

$$y(0, t) = y(100, t) = 0, \tag{2}$$

$$y(x, 0) = x^2(100 - x), \tag{3}$$

$$\frac{\partial y(x, 0)}{\partial t} = \begin{cases} x, & 0 \le x \le 25, \\ 1/3(100 - x), & 25 \le x \le 100. \end{cases} \tag{4}$$

   What is the speed of wave propagation along the string? What is the initial displacement of the string at point $x = 20$? What is the initial velocity of the string at point $x = 50$? At what point of the string is the initial velocity the largest?

9. Let us suppose that the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad x \in [0,2], \tag{5}$$
$$y(0,t) = y(\pi,t) = 0, \tag{6}$$
$$y(x,0) = 0, \tag{7}$$
$$\frac{\partial y(x,0)}{\partial t} = g(x). \tag{8}$$

Suppose that

$$\int_0^2 g(x) \sin\left(\frac{\pi n x}{2}\right)\, dx = \frac{1}{n^3}.$$

Find $y(x,t)$.

10. Let us suppose that the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad x \in [0,\pi], \tag{9}$$
$$y(0,t) = y(\pi,t) = 0, \tag{10}$$
$$y(x,0) = 22\sin 2x + 8\sin 6x, \tag{11}$$
$$\frac{\partial y(x,0)}{\partial t} = 0. \tag{12}$$

Find $y(x,t)$, in a closed form (containing no integrals). You will not need to evaluate any integrals.

11. Consider the partial differential equation,

$$\frac{\partial y}{\partial t} = 12y - 5\frac{\partial y}{\partial x} + 7\frac{\partial^2 y}{\partial x^2}.$$

Demonstrate how the *separation of variables* method works (that is, reduce this partial differential equation to some ordinary differential equations). Explain every step.

12. Suppose that we have the following problem:

$$\frac{\partial^2 y}{\partial t^2} = 2\frac{\partial^2 y}{\partial x^2}, \quad x \in [0,3], \tag{13}$$
$$y(0,t) = y(\pi,t) = 0, \tag{14}$$
$$y(x,0) = x(3-x)(2-x), \tag{15}$$
$$\frac{\partial y(x,0)}{\partial t} = 0. \tag{16}$$

Set

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{3}\right) \cos\left(\frac{n\sqrt{3}\pi t}{3}\right).$$

Which equations (13-16) does this function satisfies (for general coefficients c_n such that the series converges)? Prove your answer.

13. Suppose we have a wave equation on an infinite line,

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2}, \tag{17}$$

$$y(x,0) = f(x), \tag{18}$$

$$\frac{\partial y(x,0)}{\partial t} = 0, \tag{19}$$

where

$$f(x) = \begin{cases} x(2-x), & x \in [0,2], \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $y(x,t)$.

(b) Draw the solution for $t = 5$, and $t = 10$ (2 graphs).

(b) How long will it take before an observer located at point $x = 27$ receives the signal?

14. Suppose a bar of length $6\,cm$ has insulating ends. The initial temperature distribution is given by $f(x)$, such that

$$f(x) = \begin{cases} 2, & 0 \le x \le 3, \\ 4, & 3 < x \le 6. \end{cases}$$

(a) Write down the initial and boundary value problem for $y(x,t)$, the temperature of the bar.

(b) Write down the solution for $y(x,t)$.

(c) What is the distribution of temperature as times goes to infinity?

(d) Draw the temperature distribution for $t = 0$, for some $t_1 > 0$, for some $t_2 > t_1$ and for $t = \infty$ (4 graphs).

15. Suppose a bar of length $10\,cm$ has insulating ends. Find the temperature distribution as $t \to \infty$ if:

(a) The initial temperature distribution is given by $f(x)$, such that

$$f(x) = \begin{cases} 0, & 0 \leq x \leq 1, \\ 2, & 1 < x \leq 2, \\ 0, & 2 < x \leq 3 \\ 5, & 3 < x \leq 4, \\ 2, & 4 < x \leq 6. \end{cases}$$

(b) The initial temperature distribution is given by $f(x) = x + 2x^2$.

(c) The initial temperature distribution is given by $f(x) = x(10 - x)$, but the temperature at the ends is kept at zero (no insulation).

(d) For the problem in (c), draw the temperature distribution for $t = 0$, for some $t_1 > 0$, for some $t_2 > t_1$ and for $t = \infty$ (4 graphs).