Solutions for practice problems for the Final, part 3

Note: Practice problems for the Final Exam, part 1 and part 2 are the same as Practice problems for Midterm 1 and Midterm 2.

1. Calculate Fourier Series for the function $f(x)$, defined on $[-2, 2]$, where

$$f(x) = \begin{cases} -1, & -2 \le x \le 0, \\ 2, & 0 < x \le 2. \end{cases}$$

We have

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n x}{2} + b_n \sin \frac{\pi n x}{2} \right),$$

where

$$a_0 = \frac{1}{2} \left(\int_{-2}^{0} (-1)\, dx + \int_0^2 2\, dx \right) = 1,$$

$$a_n = \frac{1}{2} \left(\int_{-2}^{0} (-1) \cos \frac{\pi n x}{2}\, dx + \int_0^2 2 \cos \frac{\pi n x}{2}\, dx \right) =$$

$$\frac{1}{2} \left((-1) \left[\frac{2}{\pi n} \sin \frac{\pi n x}{2} \right]_{-2}^{0} + 2 \left[\frac{2}{\pi n} \sin \frac{\pi n x}{2} \right]_0^2 \right) = 0, \quad n > 0,$$

and

$$b_n = \frac{1}{2} \left(\int_{-2}^{0} (-1) \sin \frac{\pi n x}{2}\, dx + \int_0^2 2 \sin \frac{\pi n x}{2}\, dx \right) =$$

$$\frac{1}{2} \left(-(-1) \left[\frac{2}{\pi n} \cos \frac{\pi n x}{2} \right]_{-2}^{0} - 2 \left[\frac{2}{\pi n} \cos \frac{\pi n x}{2} \right]_0^2 \right) =$$

$$\frac{1}{\pi n}(1 - \cos \pi n) - 2\frac{1}{\pi n}(\cos \pi n - 1) = \frac{3}{\pi n}(1 - (-1)^n).$$

Therefore, we have

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{3}{\pi n}(1 - (-1)^n) \sin \frac{\pi n x}{2}.$$

An easy way to see that all of a_n except a_0 are zero is to note that

$$f(x) = \frac{1}{2} + g(x),$$

where $g(x)$ is an odd function,

$$g(x) = \begin{cases} 3/2, & x > 0, \\ -3/2, & x < 0. \end{cases}$$

2. Calculate Fourier Series for the function $f(x)$, defined on $[-5,5]$, where

$$f(x) = 3H(x-2).$$

By a similar method,

$$f(x) = \frac{9}{5} + \sum_{n=1}^{\infty} \left[\frac{-3}{\pi n} \sin \frac{2\pi n}{5} \cos \frac{\pi n x}{5} + \frac{3}{\pi n} \left(\cos \frac{2\pi n}{5} - (-1)^n \right) \sin \frac{\pi n x}{5} \right].$$

3. Calculate Fourier Series for the function, $f(x)$, defined as follows:

(a) $x \in [-4,4]$, and

$$f(x) = 5.$$

Comparing $f(x)$ with the general Fourier Series expression with $L = 4$,

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n x}{4} + b_n \sin \frac{\pi n x}{4} \right),$$

we can see that $a_0 = 10$, $a_n = b_n = 0$ for $n > 0$ will give $f(x) = g(x)$.

(b) $x \in [-\pi, \pi]$, and

$$f(x) = 21 + 2\sin 5x + 8\cos 2x.$$

Again, for $L = \pi$, we have

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right),$$

and setting $a_0 = 42$, $a_2 = 8$, $b_5 = 2$ and the rest of the coefficients zero, we obtain $f(x) = g(x)$.

(c) $x \in [-\pi, \pi]$, and

$$f(x) = \sum_{n=1}^{8} c_n \sin nx, \quad \text{with } c_n = 1/n.$$

Similarly, we set $b_n = 1/n$ for $1 \leq n \leq 8$, and the rest of the coefficients zero.

(d) $x \in [-3, 3]$, and

$$f(x) = -4 + \sum_{n=1}^{6} c_n(\sin(\pi n x/3) + 7\cos(\pi n x/3)), \quad \text{with } c_n = (-1)^n.$$

We have

$$g(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{\pi n x}{3} + b_n \sin \frac{\pi n x}{3} \right),$$

so we set $a_0 = -8$, $a_n = 7(-1)^n$ for $1 \leq n \leq 6$ and $b_n = (-1)^n$ for $1 \leq n \leq 6$, and the rest of the coefficients zero.

4. (a) Let $f(x) = x + x^3$ for $x \in [0, \pi]$. What coefficients of the Fourier Series of f are zero? Which ones are non-zero? Why?

   $f(x)$ is an odd function. Indeed,

$$f(-x) = -x + (-x)^3 = -x - x^3 = -(x + x^3) = -f(x),$$

   therefore $a_n = 0$, and b_n can be nonzero.

   (b) Let $g(x) = \cos(x^5) + \sin(x^2)$. What coefficients of the Fourier Series of g are zero? Which ones are non-zero? Why?

   $g(x)$ is an even function. Indeed,

$$g(-x) = \cos((-x)^5) + \sin((-x)^2) = \cos(-x^5) + \sin(x^2) = \cos(x^5) + \sin(x^2) = g(x).$$

   Therefore, $b_n = 0$, and a_n can be nonzero.

5. Let $f(x) = 2x + x^4$ for $x \in [0, 5]$.

   (a) Write down the function $G(x)$, which is the odd continuation for $f(x)$. Specify what terms will be zero and non-zero in the Fourier expansion for $G(x)$.

   We have

$$G(x) = \begin{cases} 2x + x^4, & x > 0, \\ 2x - x^4, & x < 0. \end{cases}$$

Indeed, we can check that if $\alpha > 0$, then $G(-\alpha) = -2\alpha - (-\alpha)^4 = -2\alpha - \alpha^4 = -G(\alpha)$. In the Fourier expansion for G, $a_n = 0$, and b_n can be nonzero.

(b) Write down the function $V(x)$, which is the even continuation for $f(x)$. Specify what terms will be zero and non-zero in the Fourier expansion for $V(x)$.

We have

$$V(x) = \begin{cases} 2x + x^4, & x > 0, \\ -2x + x^4, & x < 0. \end{cases}$$

Indeed, we can check that if $\alpha > 0$, then $V(-\alpha) = -2(-\alpha) + (-\alpha)^4 = 2\alpha + \alpha^4 = V(\alpha)$. In the Fourier expansion for V, $b_n = 0$, and a_n can be nonzero.

6. Suppose $f(x)$ is defined for $x \in [0, 7]$, and $f(x) = 2e^{-4x}$. Another function, $F(x)$, is given by the following:

$$F(x) = \sum_{n=0}^{\infty} a_n \cos(\pi n x/7),$$

where

$$a_n = \frac{2}{7} \int_0^7 2e^{-4x} \cos\left(\frac{\pi n x}{7}\right) dx.$$

What is the value of $F(3)$? What is the value of $F(-2)$?

The function $F(x)$ is the cosine Fourier expansion of f. On the domain of f, that is, for $x \in [0, 7]$, we have $F(x) = f(x)$. Therefore, since $3 \in [0, 7]$, then $F(3) = f(3) = 2e^{-12}$.

For the negative values of x, the cosine series converges to the even extension of $f(x)$, which is $2e^{-4|x|}$. Therefore, $F(-2) = f(2) = 2e^{-8}$.

Note: a sine Fourier series would give the odd extension, and in this case we would have $-f(2) = -2e^{-8}$.

7. Let us supposed that both ends of a string of length 25 cm are attached to fixed points at hight 0. Initially, the string is at rest, and has the shape $4\sin(2\pi x/25)$, where x is the horizontal coordinate along the string, with zero at the left end. The speed of wave propagation along

the string is $3\,cm/sec$. Write down the complete initial and boundary value problem for the shape of the string.

We have the following initial and boundary value problem:

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2}, \quad x \in [0, 25], \tag{1}$$

$$y(0, t) = y(25, t) = 0, \tag{2}$$

$$y(x, 0) = 4\sin(2\pi x/25), \tag{3}$$

$$\frac{\partial y(x, 0)}{\partial t} = 0. \tag{4}$$

8. Let us suppose that the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = 50\frac{\partial^2 y}{\partial x^2}, \quad x \in [0, 100], \tag{5}$$

$$y(0, t) = y(100, t) = 0, \tag{6}$$

$$y(x, 0) = x^2(100 - x), \tag{7}$$

$$\frac{\partial y(x, 0)}{\partial t} = \begin{cases} x, & 0 \le x \le 25, \\ 1/3(100 - x), & 25 \le x \le 100. \end{cases} \tag{8}$$

What is the speed of wave propagation along the string? What is the initial displacement of the string at point $x = 20$? What is the initial velocity of the string at point $x = 50$? At what point of the string is the initial velocity the largest?

The speed of wave propagation along the string is $\sqrt{50}$. The initial displacement of the string at point $x = 20$ is $20^2(100 - 20) = 32000$. The initial velocity of the string at point $x = 50$ is $1/3(100-50) = 50/3$. The maximum of the initial velocity is at point $x = 25$ (plot the graph of the initial velocity, equation (8), to see this).

9. Let us suppose that the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad x \in [0, 2], \tag{9}$$

$$y(0, t) = y(\pi, t) = 0, \tag{10}$$

$$y(x, 0) = 0, \tag{11}$$

$$\frac{\partial y(x, 0)}{\partial t} = g(x). \tag{12}$$

Suppose that

$$\int_0^2 g(x) \sin\left(\frac{\pi n x}{2}\right)\, dx = \frac{1}{n^3}.$$

Find $y(x,t)$.

For the problem with the zero initial displacement, the solution is given in terms of the initial velocity (here $c = 1$),

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{2} \sin\frac{n\pi t}{2},$$

with

$$c_n = \frac{2}{n\pi}\int_0^2 g(x) \sin\left(\frac{\pi n x}{2}\right)\, dx = \frac{2}{n^4\pi}.$$

10. Let us suppose that the following boundary value problem is given:

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad x \in [0,\pi], \tag{13}$$

$$y(0,t) = y(\pi,t) = 0, \tag{14}$$

$$y(x,0) = 22\sin 2x + 8\sin 6x, \tag{15}$$

$$\frac{\partial y(x,0)}{\partial t} = 0. \tag{16}$$

Find $y(x,t)$, in a closed form (containing no integrals). You will not need to evaluate any integrals.

We look for the solution is the form,

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt.$$

To satisfy initial condition (15), we set $t = 0$ and obtain,

$$y(x,0) = \sum_{n=1}^{\infty} c_n \sin nx.$$

To make this equal to $f(x) = 22\sin 2x + 8\sin 6x$, we set $c_2 = 22$, $c_6 = 8$, and the rest of them zero. We obtain,

$$y(x,t) = 22\sin 2x\cos 2t + 8\sin 6x\cos 6t.$$

11. Consider the partial differential equation,

$$\frac{\partial y}{\partial t} = 12y - 5\frac{\partial y}{\partial x} + 7\frac{\partial^2 y}{\partial x^2}.$$

Demonstrate how the *separation of variables* method works (that is, reduce this partial differential equation to some ordinary differential equations). Explain every step.

We assume that $y(x,t) = X(x)T(t)$. Then we substitute this in the equation, to obtain,

$$X\frac{\partial T}{\partial t} = 12XT - 5T\frac{\partial X}{\partial x} + 7T\frac{\partial^2 X}{\partial x^2}.$$

Now we divide both sides by XT, to get

$$\frac{\frac{\partial T}{\partial t}}{T} = 12 + \frac{-5\frac{\partial X}{\partial x} + 7\frac{\partial^2 X}{\partial x^2}}{X}.$$

The left hand side does not depend on x. The right had side does not depend on t. They are equal to each other. Therefore, both the left hand side and the right hand side are constants. We set the constant to be $-\mu$, and obtain 2 ordinary differential equations:

$$\frac{\partial T}{\partial t} = -\mu T,$$

$$-5\frac{\partial X}{\partial x} + 7\frac{\partial^2 X}{\partial x^2} = (-\mu - 12)X.$$

12. Suppose that we have the following problem:

$$\frac{\partial^2 y}{\partial t^2} = 2\frac{\partial^2 y}{\partial x^2}, \quad x \in [0,3], \tag{17}$$

$$y(0,t) = y(\pi,t) = 0, \tag{18}$$

$$y(x,0) = x(3-x)(2-x), \tag{19}$$

$$\frac{\partial y(x,0)}{\partial t} = 0. \tag{20}$$

Set

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{3}\right)\cos\left(\frac{n\sqrt{3}\pi t}{3}\right).$$

Which equations (17-20) does this function satisfies (for general coefficients c_n such that the series converges)? Prove your answer.

Please not there is an error, it has to be $\sqrt{2}$ in the solution, not $\sqrt{3}$. We have

$$y(x,t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{3}\right) \cos\left(\frac{n\sqrt{2}\pi t}{3}\right).$$

The function $y(x,t)$ satisfies equation (17), because each of the additives, $\sin\left(\frac{n\pi x}{3}\right)\cos\left(\frac{n\sqrt{2}\pi t}{3}\right)$, satisfies it. Indeed, let us denote $p_n = \sin\left(\frac{n\pi x}{3}\right)\cos\left(\frac{n\sqrt{2}\pi t}{3}\right)$. We have

$$\frac{\partial^2 p_n}{\partial t^2} = -n^2 2\pi^2/9 p_n,$$

and

$$\frac{\partial^2 p_n}{\partial x^2} = -n^2\pi^2/9 p_n,$$

so

$$\frac{\partial^2 p_n}{\partial t^2} = 2\frac{\partial^2 p_n}{\partial x^2}.$$

Also, each term p_n satisfies the boundary conditions, equation (18). Indeed,

$$p_n(0,t) = \sin\left(\frac{n\pi 0}{3}\right)\cos\left(\frac{n\sqrt{2}\pi t}{3}\right) = 0, \quad p_n(3) = \sin(n\pi)\cos\left(\frac{n\sqrt{2}\pi t}{3}\right) = 0.$$

Finally, it satisfies condition (20):

$$\frac{\partial p_n(x,0)}{\partial t} = \sin\left(\frac{n\pi x}{3}\right)\frac{-n\sqrt{2}\pi}{3}\sin\left(\frac{n\sqrt{2}\pi 0}{3}\right) = 0.$$

It does not satisfy condition (19), unless we choose the coefficients, c_n, in a clever way.

13. Suppose we have a wave equation on an infinite line,

$$\frac{\partial^2 y}{\partial t^2} = 9\frac{\partial^2 y}{\partial x^2}, \tag{21}$$

$$y(x,0) = f(x), \tag{22}$$

$$\frac{\partial y(x,0)}{\partial t} = 0, \tag{23}$$

where

$$f(x) = \begin{cases} x(2-x), & x \in [0,2], \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $y(x,t)$.

The D'Alembert's solution is:

$$y(x,t) = \frac{1}{2}(f(x-3t) + f(x+3t)).$$

(b) Draw the solution for $t = 5$, and $t = 10$ (2 graphs).

The graphs will contain two tent-shaped signals, with base of length 2, of height 1, centered at $x = -4$ and $x = 6$ for $t = 5$, and at $x = -9$ and $x = 11$ for $t = 10$.

(c) How long will it take before an observer located at point $x = 27$ receives the signal?

The right side of the "tent" will reach $x = 27$ after time $(27-2)/3 = 25/3$.

14. Suppose a bar of length $6\,cm$ has insulating ends. The initial temperature distribution is given by $f(x)$, such that

$$f(x) = \begin{cases} 2, & 0 \le x \le 3, \\ 4, & 3 < x \le 6. \end{cases}$$

(a) Write down the initial and boundary value problem for $y(x,t)$, the temperature of the bar.

We have the system,

$$\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial x^2}, \quad x \in [0,6], \tag{24}$$

$$\frac{\partial y(0,t)}{\partial x} = \frac{\partial y(6,t)}{\partial x} = 0, \tag{25}$$

$$y(x,0) = f(x). \tag{26}$$

(b) Write down the solution for $y(x,t)$.

We have

$$y(x,t) = \frac{c_0}{2} + \sum_{c=1}^{\infty} c_n \cos\frac{n\pi x}{6} e^{-k\left(\frac{n\pi}{6}\right)^2 t},$$

where

$$c_n = \frac{2}{6}\int_0^6 f(x)\cos\frac{n\pi x}{6}\,dx = \frac{-4}{n\pi}\sin\frac{n\pi}{2}, \quad n > 0,$$

and $c_0 = 6$.

(c) What is the distribution of temperature as times goes to infinity?

The equilibrium temperature is uniformly distributed at the level

$$\frac{c_0}{2} = 3.$$

(d) Draw the temperature distribution for $t = 0$, for some $t_1 > 0$, for some $t_2 > t_1$ and for $t = \infty$ (4 graphs).

See lecture notes.

15. Suppose a bar of length $10\,cm$ has insulating ends. Find the temperature distribution as $t \to \infty$ if:

(a) The initial temperature distribution is given by $f(x)$, such that

$$f(x) = \begin{cases} 0, & 0 \le x \le 1, \\ 2, & 1 < x \le 2, \\ 0, & 2 < x \le 3 \\ 5, & 3 < x \le 4, \\ 2, & 4 < x \le 6. \end{cases}$$

Note: there is a misprint: in the last inequality we take $4 < x < 10$, because the length is 10, not 6. We compute the limiting temperature level as the average of f,

$$\frac{0 + 2 + 0 + 5 + 2 \cdot 6}{10} = 19/10.$$

(b) The initial temperature distribution is given by $f(x) = x + 2x^2$.

Similarly, compute the average,

$$\frac{1}{10}\int_0^{10} f(x)\,dx = \frac{1}{10}\int_0^{10} x(10-x)\,dx = 50/3.$$

(c) The initial temperature distribution is given by $f(x) = x(10-x)$, but the temperature at the ends is kept at zero (no insulation).

In this case, the temperature decays to zero, since energy is allowed to leave the system.

(d) For the problem in (c), draw the temperature distribution for $t = 0$, for some $t_1 > 0$, for some $t_2 > t_1$ and for $t = \infty$ (4 graphs).

See lecture notes.