

## Math 421 Midterm #1 Solutions

1. Function  $f(t)$  is defined as follows:

$$f(t) = \begin{cases} 0, & t < 1, \\ t + 1, & 1 \leq t \leq 2, \\ e^{-t}, & t > 2. \end{cases}$$

(a) Sketch the function  $f(t)$ .

(b) Express  $f(t)$  in terms of the Heaviside functions.

$$f(t) = [H(t-1) - H(t-2)](t+1) + H(t-2)e^{-t}.$$

(c) Calculate the Laplace transform of  $f(t)$ .

$$\mathcal{L}[f] = \mathcal{L}[H(t-1)(t+1)] - \mathcal{L}[H(t-2)(t+1)] + \mathcal{L}[H(t-2)e^{-t}].$$

$$\mathcal{L}[H(t-1)(t+1)] = \mathcal{L}[H(t-1)(t-1+2)] = \mathcal{L}[H(t-1)(t-1)] + 2\mathcal{L}[H(t-1)] = e^{-s}\frac{1}{s^2} + 2e^{-s}\frac{1}{s}.$$

Similarly,

$$\mathcal{L}[H(t-2)(t+1)] = e^{-2s}\frac{1}{s^2} + 3e^{-2s}\frac{1}{s}.$$

Finally,

$$\mathcal{L}[H(t-2)e^{-t}] = \frac{e^{-2(s+1)}}{s+1},$$

by 1st shifting theorem. All together,

$$\mathcal{L}[f] = e^{-s}\frac{1}{s^2} + 2e^{-s}\frac{1}{s} - e^{-2s}\frac{1}{s^2} - 3e^{-2s}\frac{1}{s} + \frac{e^{-2(s+1)}}{s+1}.$$

2. Find the inverse Laplace transform of the functions

$$F(s) = \frac{1}{1-s^2}, \quad \text{and} \quad G(s) = \frac{s}{1-s^2}.$$

By partial fractions,

$$\mathcal{L}^{-1} \left[ \frac{1}{(1-s)(1+s)} \right] = \frac{1}{2} \mathcal{L}^{-1} \left[ \frac{1}{1-s} + \frac{1}{1+s} \right] = \frac{1}{2} (-e^t + e^{-t}) = f(t).$$

Similarly,

$$\mathcal{L}^{-1} \left[ \frac{2}{(1-s)(1+s)} \right] = \frac{1}{2} \mathcal{L}^{-1} \left[ \frac{1}{1-s} - \frac{1}{1+s} \right] = \frac{1}{2} (-e^t - e^{-t}) = g(t).$$

(Note that  $f(t) = -\sinh t$  and  $g(t) = -\cosh t$ .)

3. Calculate

$$\int_0^{\infty} \left( \delta(t - \pi) \sin \frac{t}{2} + 4H(t - 3)\delta(t - 7) \right) dt.$$

This equals

$$\sin \frac{\pi}{2} + 4H(7 - 3) = 1 + 4 = 5.$$

4. Define  $f(t) = H(t)e^{-2t}$  and  $g(t) = t + 1$ .

(a) Compute  $F(t) = f * g$ .

(b) Find the Laplace transform of  $F(t)$ .

The convolution

$$\begin{aligned} f * g &= \int_0^t f(t-\tau)g(\tau) d\tau = \int_0^t H(t-\tau)e^{-2(t-\tau)}(\tau+1) d\tau = e^{-2t} \int_0^t e^{2\tau} H(t-\tau)(\tau+1) d\tau = \\ &e^{-2t} \int_0^t e^{2\tau}(\tau+1) d\tau. \end{aligned}$$

This is because  $H(t-\tau) = 1$  for all  $0 \leq \tau \leq t$ . We continue,

$$e^{-2t} \left( \int_0^t e^{2\tau} \tau d\tau + \int_0^t e^{2\tau} d\tau \right).$$

The first integral is taken by parts,

$$\int_0^t e^{2\tau} \tau d\tau = \tau \frac{e^{2\tau}}{2} \Big|_0^t - \int_0^t \frac{e^{2\tau}}{2} d\tau = \frac{1}{2} t e^{2t} - \frac{1}{4} (e^{2t} - 1).$$

The second integral is evaluated,

$$\int_0^t e^{2\tau} d\tau = \frac{1}{2} (e^{2t} - 1).$$

Together, we get

$$e^{-2t} \left( \frac{t}{2} e^{2t} + \frac{1}{4} e^{2t} - \frac{1}{4} \right) = \frac{t}{2} + \frac{1}{4} - \frac{1}{4} e^{-2t}.$$

The Laplace transform can be calculated directly,

$$\mathcal{L}[F] = \frac{1}{2s^2} + \frac{1}{4s} - \frac{1}{4(s+2)}.$$

Alternatively, this is the product of the Laplace transforms of the parts,

$$\mathcal{L}[F] = \mathcal{L}[H(t)e^{-2t}] \mathcal{L}[t+1] = \frac{1}{s+2} \left( \frac{1}{s^2} + \frac{1}{s} \right).$$

These two expressions are the same.

5. Solve by using Laplace transform,

$$y'' + 5y' + 6y = H(t - 10) + \delta(t - 20), \quad y(0) = 0, \quad y'(0) = 1.$$

$$s^2Y - sy(0) - y'(0) + 5(sY - y(0)) + 6Y = \frac{e^{-10s}}{s} + e^{-20s}.$$

$$s^2Y - 1 + 5sY + 6Y = \frac{e^{-10s}}{s} + e^{-20s}.$$

$$Y(s^2 + 5s + 6) = \frac{e^{-10s}}{s} + e^{-20s} + 1.$$

$$Y(s) = \frac{e^{-10s}}{s(s+2)(s+3)} + \frac{e^{-20s}}{(s+2)(s+3)} + \frac{1}{(s+2)(s+3)}.$$

Now we take the inverse Laplace. First we have,

$$\mathcal{L}^{-1} \left[ \frac{1}{(s+2)(s+3)} \right] = e^{-2t} - e^{-3t} = f(t).$$

Therefore,

$$\mathcal{L}^{-1} \left[ \frac{e^{-20s}}{(s+2)(s+3)} \right] = H(t-20)f(t-20).$$

Also, we need to find

$$\mathcal{L}^{-1} \left[ \frac{1}{s(s+2)(s+3)} \right] = 1 * f(t) = \int_0^t (e^{-2t} - e^{-3t}) dt = \frac{1}{2}(1 - e^{-2t}) - \frac{1}{3}(1 - e^{-3t}) =$$

$$\frac{1}{6} (1 - 3e^{-2t} + 2e^{-3t}).$$

Therefore,

$$\mathcal{L}^{-1} \left[ \frac{e^{-10s}}{s(s+2)(s+3)} \right] = H(t-10) \frac{1}{6} (1 - 3e^{-2(t-10)} + 2e^{-3(t-10)}).$$

All together,

$$y(t) = e^{-2t} - e^{-3t} + H(t-20) (e^{-2(t-20)} - e^{-3(t-20)}) + H(t-10) \frac{1}{6} (1 - 3e^{-2(t-10)} + 2e^{-3(t-10)}).$$

6. Solve by using Laplace transform,

$$y_1' + y_2 = 2\delta(t-4), \quad y_1 + y_2' = 0, \quad y_1(0) = 0, \quad y_2(0) = 1.$$

We have

$$sY_1 + Y_2 = 2e^{-4s}, \quad Y_1 + sY_2 - 1 = 0.$$

From the second equation,

$$Y_1 = 1 - sY_2,$$

and substituting this into the first equation, we get

$$s(1 - sY_2) + Y_2 = 2e^{-4s},$$

which gives,

$$Y_2 = \frac{2e^{-4s}}{1 - s^2} - \frac{s}{1 - s^2}.$$

Therefore, the expression for  $Y_1$  is

$$Y_1 = \frac{1}{1 - s^2} - \frac{2se^{-4s}}{1 - s^2}.$$

Using the results of problem 2, we have

$$\mathcal{L}^{-1}\left[\frac{1}{(1-s)(1+s)}\right] = f(t), \quad \mathcal{L}^{-1}\left[\frac{2}{(1-s)(1+s)}\right] = g(t).$$

We have,

$$\mathcal{L}^{-1}\left[\frac{2e^{-4s}}{1 - s^2}\right] = 2H(t-4)f(t-4) = H(t-4)(-e^{t-4} + e^{-(t-4)}).$$

$$\mathcal{L}^{-1}\left[\frac{2se^{-4s}}{1 - s^2}\right] = 2H(t-4)g(t-4) = H(t-4)(-e^{t-4} - e^{-(t-4)}).$$

All together,

$$y_1(t) = \frac{1}{2}(-e^t + e^{-t}) - H(t-4)(-e^{t-4} - e^{-(t-4)}).$$

$$y_2(t) = -\frac{1}{2}(-e^t - e^{-t}) + H(t-4)(-e^{t-4} + e^{-(t-4)}).$$

7. Suppose

$$\mathbf{F}_1 = (0, 1, 3, 2, 7), \quad \mathbf{F}_2 = (2, 1, 2, 4, 7), \quad \mathbf{F}_3 = (2, -1, -4, 0, -7).$$

Let the subspace,  $V$ , be spanned by  $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$ .

- (a) Explain why  $\mathbf{F}_1, \mathbf{F}_2$  and  $\mathbf{F}_3$  are not a basis in  $V$ .
- (b) Find a basis for  $V$ .
- (c) What is the dimension of  $V$ ?
- (d) Give an example of a vector, different from  $\mathbf{F}_1, \mathbf{F}_2$  and  $\mathbf{F}_3$ , which belongs to  $V$ .

(a) In order to be a basis of a subspace,  $V$ , vectors  $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3$  must (i) span  $V$  and (ii) be linearly independent. We check condition (ii):

$$\alpha\mathbf{F}_1 + \beta\mathbf{F}_2 + \gamma\mathbf{F}_3 = \mathbf{0}.$$

$$(2\beta + 2\gamma, \alpha + \beta - \gamma, 3\alpha + 2\beta - 4\gamma, 2\alpha + 4\beta, 7\alpha + 7\beta - 7\gamma) = (0, 0, 0, 0, 0).$$

$$2\beta + 2\gamma = 0,$$

$$\alpha + \beta - \gamma = 0,$$

$$3\alpha + 2\beta - 4\gamma = 0,$$

$$2\alpha + 4\beta = 0,$$

$$7\alpha + 7\beta - 7\gamma = 0.$$

From the first equation,  $\beta = -\gamma$ . From the 5th equation,  $7(\alpha + \beta - \gamma) = 7(\alpha - 2\gamma) = 0$ , which means that  $\alpha = 2\gamma$ . Let us take  $\gamma = 1$ , which makes  $\beta = -1$  and  $\alpha = 2$ . These constants satisfy the above system of equations. They are not simultaneously zero. Therefore, the vectors  $\mathbf{F}_1, \mathbf{F}_2$  and  $\mathbf{F}_3$  are not linearly independent. Thus, they cannot serve as a basis.

(b) Since the vectors are not independent, one of them can be expressed as a linear combination of the others, e.g.  $\mathbf{F}_3 = \frac{\alpha}{\gamma}\mathbf{F}_1 + \frac{\beta}{\gamma}\mathbf{F}_2 = 2\mathbf{F}_1 - \mathbf{F}_2$ . The other two vectors,  $\mathbf{F}_1$  and  $\mathbf{F}_2$  are linearly independent, and they span  $V$ . Therefore, they can be a basis for  $V$ .

(c) There are two vectors in the basis, so the dimension is 2.

(d) One example is  $(0, 0, 0, 0, 0)$ . Another example is  $\mathbf{F}_1 + \mathbf{F}_3 = (2, 0, -1, 2, 0)$ .