Practice problems for Midterm 2

1. Suppose that A is a 100×100 matrix with all elements equal to zero except for the diagonal elements, $a_{ii} = 3$.

   (a) Calculate $|A|$.

   (b) Suppose that the matrix B is the same a A, except is has one more nonzero element: $b_{13,74} = 5$. Calculate $|B|$.

   (c) Suppose that the matrix C is the same a B, except is has one more nonzero element: $c_{81,80} = 7$. Calculate $|C|$.

2. Suppose that the matrix A is given by

$$A = \begin{pmatrix} 1 & 4 & 3 & 1 \\ 11 & 16 & 9 & 11 \\ 2 & 1 & 0 & 2 \\ -1 & 10 & 9 & -1 \end{pmatrix}.$$

   (a) Does the system $AX = 0$ have non-zero solutions? Why?

   (b) How many independent variables does the system $AX = 0$ have? How many dependent variables? Find all the solutions of the system $AX = 0$.

   (c) Suppose we have a non-homogeneous system $AX = B$, where

$$B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

   Specify the conditions on the components of B which would guarantee that the equation $AX = B$ has non-zero solutions.

   (d) Find a specific vector, B, for which the system has no solutions.

   (e) Find a specific vector, B, for which the system has at least one solution. Find all solutions of this system.

   Use Row Reduction Facts.

3. Suppose A is a 50×70 matrix with rank 40.

   (a) Does the system $AX = 0$ have non-zero solutions?

   (b) How many independent variables does the system $AX = 0$ have? How many dependent variables?

   (c) Suppose we have the non-homogeneous system $AX = B$. Describe the procedure by which you determine whether this system has non-zero solutions for a given right hand side, B. This procedure will be solving a certain system of equations, other than the system $AX = 0$. How many equations will you have to solve? For how many variables?

4. Suppose that the matrix A is given by

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & x & 0 & 2 \\ 2 & 0 & 3 & 4 \\ 3 & 0 & 0 & x \end{pmatrix}.$$

   For which values of x is this matrix singular?

5. Suppose we have vectors,

$$F_1 = (-2, -4, 8, -7), \quad F_2 = (1, 2, 1, 1), \quad F_3 = (1, 2, -1, 2).$$

   Are they linearly independent?

   Use Row Reduction Facts.

6. Solve the system $AX = F$ where

$$A = \begin{pmatrix} 1 & -4 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -5 & 6 \end{pmatrix},$$

   and $F = (0, 2, 1)$. Use Row Reduction Facts.

7. Prove that the vectors $(1, 4, 2)$, $(2, 7, 4)$, $(0, 1, -2)$ and $(1, -7, -6)$ are linearly dependent. Use an argument based on dimension.

8. Suppose that $\det A = 3$. Compute $\det A^3$, $\det(2A)$, $\det A^{-1}$ and $\det AA^T$.

9. Suppose that the inverse of A is given by

$$\begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 3 & 0 & 3 \\ 1 & -3 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

(a) Find all solution of the system $AX = B$ with $B = (1, 0, 2, 0)$.

(b) Find all solutions of the system $AX = 0$.

10. Consider vectors $(1, 2, 1)$, $(-1, -3, -2)$ and $(2, 1, 1)$.

(a) Explain why these vectors are a basis in $\mathbf{R}^3$

(b) Write a linear combination of these vectors whose sum is equal to the vector $(1, 0, 0)$.

(c) Is $(2, 5, 3)$ in the span of these vectors?

Use Row Reduction Facts.

11. Explain why $A = \begin{pmatrix} 0 & 0 \\ 18 & 0 \end{pmatrix}$ cannot be diagonalized.

12. Solve problems 9 and 11 of Section 8.2.

13. Review the definitions of the following terms: homogeneous, inhomogeneous, basis, linear independence, linear combination, subspace, dimension, span, rank, eigenvalue, eigenvector, transpose, diagonalization.

Useful row reduction facts

$$\left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 2 & 5 & 4 & -2 & b_2 \\ 0 & 1 & -4 & 6 & b_3 \\ -1 & -4 & 4 & -8 & b_4 \end{array}\right) \rightarrow \left(\begin{array}{cccc|c} 1 & 0 & 12 & -16 & -4b_3 - b_4 \\ 0 & 1 & -4 & 6 & b_3 \\ 0 & 0 & 0 & 0 & b_1 + b_3 + b_4 \\ 0 & 0 & 0 & 0 & b_2 + 3b_3 + 2b_4 \end{array}\right).$$

$$\left(\begin{array}{ccccc|c} 1 & -4 & 0 & 0 & 1 & f_1 \\ 0 & 2 & 0 & -1 & 0 & f_2 \\ 0 & 1 & 0 & -5 & 6 & f_3 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -5/3 & 1/9(9f_1+20f_2-4f_3) \\ 0 & 1 & 0 & 0 & -2/3 & 1/9(5f_2-f_3) \\ 0 & 0 & 0 & 1 & -4/3 & 1/9(f_2-2f_3) \end{array}\right).$$

$$\begin{pmatrix} 1 & 4 & 2 \\ 2 & 5 & -1 \\ -4 & -7 & 7 \\ 7 & 25 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -14/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 3 & 1 & b_1 \\ 11 & 16 & 9 & 11 & b_2 \\ 2 & 1 & 0 & 2 & b_3 \\ -1 & 10 & 9 & -1 & b_4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -3/7 & 1 & 1/21(10b_3-b_4) \\ 0 & 1 & 6/7 & 0 & 1/21(b_3+2b_4) \\ 0 & 0 & 0 & 0 & 1/3(3b_1-2b_3-b_4) \\ 0 & 0 & 0 & 0 & b_2-6b_3-b_4 \end{array}\right).$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & f_1 \\ 2 & -3 & 1 & f_2 \\ 1 & -2 & 1 & f_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/2(f_1+3f_2-5f_3) \\ 0 & 1 & 0 & 1/2(f_1+f_2-3f_3) \\ 0 & 0 & 1 & 1/2(f_1-f_2+f_3) \end{array}\right).$$

$$\left(\begin{array}{ccccc|c} 1 & 4 & -1 & 4 & 0 & f_1 \\ 3 & 1 & -1 & 2 & 3 & f_2 \\ 0 & 2 & -4 & 3 & -1 & f_3 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 11/40 & 9/8 & 1/40(-2f_1+14f_2-3f_3) \\ 0 & 1 & 0 & 17/20 & -1/4 & 1/20(6f_1-2f_2-f_3) \\ 0 & 0 & 1 & -13/40 & 1/8 & 1/40(6f_1-2f_2-11f_3) \end{array}\right).$$

$$\begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 8 & 1 & -1 \\ -7 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$