Practice problems for Midterm 2

1. Suppose that A is a 100×100 matrix with all elements equal to zero except for the diagonal elements, $a_{ii} = 3$.

   (a) Calculate $|A|$.

   $|A| = 3^{100}$, because B is a diagonal matrix.

   (b) Suppose that the matrix B is the same a A, except is has one more nonzero element: $b_{13,74} = 5$. Calculate $|B|$.

   $|B| = 3^{100}$, because B is an upper triangular matrix.

   (c) Suppose that the matrix C is the same a B, except is has one more nonzero element: $c_{81,80} = 7$. Calculate $|C|$.

   Let us expand C in terms of the co-factors of row 81. This expansion contains only two terms,

   $$|C| = c_{81,81}(-1)^{81+81}M_{81,81} + c_{81,80}(-1)^{81+80}M_{81,80}.$$

   We notice that the first term can be evaluated: $M_{81,81}$ is the determinant of an upper triangular matrix (by crossing out the 81th row, we get rid of the nonzero element in the lower part of the matrix). Therefore, $M_{81,81} = 3^{99}$. On the other hand, $M_{81,80} = 0$ because this is the determinant of a matrix which has a row (the 80th row) consisting entirely of zeros. Therefore, we have, $|C| = 3^{100}$.

2. Suppose that the matrix A is given by

   $$A = \begin{pmatrix} 1 & 4 & 3 & 1 \\ 11 & 16 & 9 & 11 \\ 2 & 1 & 0 & 2 \\ -1 & 10 & 9 & -1 \end{pmatrix}.$$

   (a) Does the system $AX = 0$ have non-zero solutions? Why?

   Yes. We find the reduced form of A among the "Useful reduction facts", (d). We can see that Rank$A = 2$, and the size of A is $n \times m$, with $n = m = 4$, so $m >$Rank$A$, and thus the homogeneous equation must have non-zero solutions.

(b) How many independent variables does the system $AX = 0$ have? How many dependent variables? Find all the solutions of the system $AX = 0$.

$m - \text{Rank}A = 2$ independent variables, and 2 dependent variables. We have

$$x_1 = 3/7x_3 - x_4, \quad x_2 = -6/7x_3.$$

(c) Suppose we have a non-homogeneous system $AX = B$, where

$$B = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}.$$

Specify the conditions on the components of B which would guarantee that the equation $AX = B$ has non-zero solutions.

In order for the equation $AX = B$ to have solutions, we need to have $\text{Rank}A = \text{Rank}(A|B)$. Thus the last two entries of the reduced right hand side must be zero. This gives the following conditions:

$$3b_1 - 2b_3 - b_4 = 0, \quad b_2 - 6b_3 - b_4 = 0.$$

(d) Find a specific vector, B, for which the system has no solutions.

$$b_1 = 1, \quad b_2 = 1, \quad b_3 = 0, \quad b_4 = 0$$

do not satisfy the conditions above.

(e) Find a specific vector, B, for which the system has at least one solution. Find all solutions of this system.

Take $b_3 = 1$, $b_4 = 0$ and find from the conditions in (c):

$$b_1 = 2, \quad b_2 = 6.$$

Now we cab write down the solution using the reduced form:

$$x_1 = 3/7x_3 - x_4 + 10/21, \quad x_2 = -6/7x_3 + 1/21.$$

3. Suppose A is a 50×70 matrix with rank 40.

(a) Does the system $AX = 0$ have non-zero solutions?

Yes. Rank$A = 40$, and the size of A is $n \times m$, with $n = 50$ and $m = 70$, so $m >$Rank$A$, and thus the homogeneous equation must have non-zero solutions.

(b) How many independent variables does the system $AX = 0$ have? How many dependent variables?

$m -$ Rank$A = 30$ independent variables, and $n - 30 = 20$ dependent variables.

(c) Suppose we have the non-homogeneous system $AX = B$. Describe the procedure by which you determine whether this system has non-zero solutions for a given right hand side, B. This procedure will be solving a certain system of equations, other than the system $AX = 0$. How many equations will you have to solve? For how many variables?

We will have to find the reduced form of A and of $A|B$. We need to have Rank$(A|B)$ =Rank$A = 40$, so the last $50 - 40 = 10$ entries of the reduced right hand side must be zero. This gives 10 equations for 50 variables (the total number of entries in the right hand side, $b_1, \ldots, b_{50}$.

4. Suppose that the matrix A is given by

$$A = \begin{pmatrix} 2 & 3 & 4 & 5 \\ 0 & x & 0 & 2 \\ 2 & 0 & 3 & 4 \\ 3 & 0 & 0 & x \end{pmatrix}.$$

For which values of x is this matrix singular?

We have $|A| = 54 + 3x - 2x^2$. This expression is equal to zero for the values $x = -9/2$, $x = 6$, and therefore for these values the matrix is singular.

5. Suppose we have vectors,

$$F_1 = (-2, -4, 8, -7), \quad F_2 = (1, 2, 1, 1), \quad F_3 = (1, 2, -1, 2).$$

Are they linearly independent?

In order to see whether these vectors are linearly independent, we form a linear combination,

$$x_1F_1 + x_2F_2 + x_3F_3,$$

where x_1, x_2 and x_3 are some constants. If there exist values x_1, x_2 and x_3 (not all equal to zero) such that the linear combination above is equal to zero, then the vectors are not linearly independent. Let us write the expression above in components,

$$(-2x_1+x_2+x_3, -4x_1+2x_2+2x_3, 8x_1+x_2-x_3, -7x_1+x_2+2x_3) = (0, 0, 0, 0).$$

This is equivalent to a homogeneous system of equations, $AX = 0$, with

$$A = \begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 8 & 1 & -1 \\ -7 & 1 & 2 \end{pmatrix}.$$

Does it have non-zero solutions? From the reduction facts (g) we see that $\mathrm{Rank}A = 2$ and this is $< m$, because $m = 4$, so that $\mathrm{Rank}A < m$. Therefore, there are nonzero solutions. Therefore, the vectors are NOT linearly independent.

6. Solve the system $AX = F$ where

$$A = \begin{pmatrix} 1 & -4 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & -5 & 6 \end{pmatrix},$$

and $F = (0, 2, 1)$. Use Row Reduction Facts.

Solving this system is equivalent to solving the reduced system, where the right hand side is given by

$$\begin{pmatrix} 1/9(9f_1 + 20f_2 - 4f_3) \\ 1/9(5f_2 - f_3) \\ 1/9(f_2 - 2f_3) \end{pmatrix},$$

see useful facts (b). Using the given values $f_1 = 0$, $f_2 = 2$ and $f_3 = 1$, we obtain that the reduce right hand side is $(4, 1, 0)^T$, and

$$x_1 = 4 + 5x_5/3, \quad x_2 = 1 + 2x_5/3, \quad x_4 = 4x_5/3.$$

7. Prove that the vectors $(1,4,2)$, $(2,7,4)$, $(0,1,-2)$ and $(1,-7,-6)$ are linearly dependent. Use an argument based on dimension.

   These four vectors belong to $\mathbf{R}^3$, the usual, "physical", three-dimensional space. Let us suppose that they are linearly independent. Then, they must be a basis in $\mathbf{R}^3$. This means that the space that they span is 4-dimensional, which is a contradiction (it belongs to a 3-dimensional space, so its dimension must be 3 or lower). Therefore, the four vectors must be linearly dependent.

   Another way to show this is two form a linear combination, like in Problem 5. This will lead to 3 equations for 5 unknown. Such a system always has non-zero solutions, because $\text{Rank}A \leq 3$ (the rank cannot be bigger than the number of lines, $n = 3$). Therefore, $\text{Rank}A < m$ because $m = 5$. Therefore, the system must have non-zero solutions. Therefore, the vectors are not linearly independent.

8. Suppose that $\text{det}A = 3$. Compute $\text{det}A^3$, $\text{det}A^{-1}$ and $\text{det}AA^T$.

$$\text{det}A^3 = 3^3 = 27, \quad \text{det}A^{-1} = 3^{-1},$$
$$\text{det}AA^T = \text{det}A det A^T = \text{det}A\text{det}A = 3^2 = 9.$$

   (see identities for determinants, Theorems 7.4, 7.4 etc)

9. Suppose that the inverse of A is given by

$$\begin{pmatrix} 1 & -1 & 2 & -2 \\ 0 & 3 & 0 & 3 \\ 1 & -3 & 2 & 1 \\ 1 & 1 & 0 & 1 \end{pmatrix}.$$

   (a) Find all solution of the system $AX = B$ with $B = (1,0,2,0)$.

   We have, for square nonsingular matrices, $X = A^{-1}B$, so we perform the multiplication to get $X = (5,0,5,1)^T$.

   (b) Find all solutions of the system $AX = 0$.

   Since the matrix A is non-singular (it has an inverse!), then the reduce matrix, $A_R = I_4$ (the identity matrix), and the only solution is the zero solution, $X = (0,0,0,0)^T$.

10. Consider vectors $(1,2,1)$, $(-1,-3,-2)$ and $(2,1,1)$.

(a) Explain why these vectors are a basis in $\mathbf{R}^3$

Let us set $V_1 = (1,2,1)$, $V_2 = (-1,-3,-2)$ and $V_3 = (2,1,1)$. We need to show that (i) these vectors span $\mathbf{R}^3$ and that (ii) they are linearly independent. To show (i), we need to prove that for any vector, $F = (f_1, f_2, f_3)$ in $\mathbf{R}^3$, we can find a linear combination, $x_1V_1 + x_2V_2 + x_3V_3 = F$. This is equivalent to solving a non-homogeneous linear system $AX = F$ with

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & -3 & 1 \\ 1 & -2 & 1 \end{pmatrix}.$$

Using the Reduction facts (e), we can see that Rank$A = 3$, and this system has solutions. To prove (ii), it is enough to show that there is no such numbers, x_1, x_2 and x_3 (not equal to zero simultaneously) such that $x_1V_1 + x_2V_2 + x_3V_3 = 0$. Since Rank$A = 3$, we can see that the only solution of the homogeneous system is the zero solution, which means that the vectors are linearly independent.

(b) Write a linear combination of these vectors whose sum is equal to the vector $(1,0,0)$.

We need to solve the system $AX = F$ for $F = (1,0,0)^T$. We have, $x_1 = x_2 = x_3 = 1/2$.

(c) Is $(2,5,3)$ in the span of these vectors?

Yes, it is, like any other vector in $\mathbf{R}^3$.

11. Explain why $A = \begin{pmatrix} 0 & 0 \\ 18 & 0 \end{pmatrix}$ cannot be diagonalized. Let us find the eigenvalues and eigenvectors of A. We have,

$$\det \begin{pmatrix} 0-\lambda & 0 \\ 18 & 0-\lambda \end{pmatrix} = \lambda^2.$$

$|A| = 0$ is equivalent to $\lambda^2 = 0$, so the repeated eigenvalue is $\lambda = 0$. The corresponding eigenvector is $\alpha(0,1)^T$. We do not have two linearly independent eigenvectors. Therefore, the matrix cannot be diagonalized.

12. Solve problems 9 and 11 of Section 8.2.

13. Review the definitions of the following terms: homogeneous, inhomogeneous, basis, linear independence, linear combination, subspace, dimension, span, rank, eigenvalue, eigenvector, transpose, diagonalization.

Useful row reduction facts

$$(a)\quad \left(\begin{array}{cccc|c} 1 & 3 & 0 & 2 & b_1 \\ 2 & 5 & 4 & -2 & b_2 \\ 0 & 1 & -4 & 6 & b_3 \\ -1 & -4 & 4 & -8 & b_4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & 12 & -16 & -4b_3 - b_4 \\ 0 & 1 & -4 & 6 & b_3 \\ 0 & 0 & 0 & 0 & b_1 + b_3 + b_4 \\ 0 & 0 & 0 & 0 & b_2 + 3b_3 + 2b_4 \end{array}\right).$$

$$(b)\quad \left(\begin{array}{ccccc|c} 1 & -4 & 0 & 0 & 1 & f_1 \\ 0 & 2 & 0 & -1 & 0 & f_2 \\ 0 & 1 & 0 & -5 & 6 & f_3 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -5/3 & 1/9(9f_1 + 20f_2 - 4f_3) \\ 0 & 1 & 0 & 0 & -2/3 & 1/9(5f_2 - f_3) \\ 0 & 0 & 0 & 1 & -4/3 & 1/9(f_2 - 2f_3) \end{array}\right).$$

$$(c)\quad \begin{pmatrix} 1 & 4 & 2 \\ 2 & 5 & -1 \\ -4 & -7 & 7 \\ 7 & 25 & 9 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -14/3 \\ 0 & 1 & 5/3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$(d)\quad \left(\begin{array}{cccc|c} 1 & 4 & 3 & 1 & b_1 \\ 11 & 16 & 9 & 11 & b_2 \\ 2 & 1 & 0 & 2 & b_3 \\ -1 & 10 & 9 & -1 & b_4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 0 & -3/7 & 1 & 1/21(10b_3 - b_4) \\ 0 & 1 & 6/7 & 0 & 1/21(b_3 + 2b_4) \\ 0 & 0 & 0 & 0 & 1/3(3b_1 - 2b_3 - b_4) \\ 0 & 0 & 0 & 0 & b_2 - 6b_3 - b_4 \end{array}\right).$$

$$(e)\quad \left(\begin{array}{ccc|c} 1 & -1 & 2 & f_1 \\ 2 & -3 & 1 & f_2 \\ 1 & -2 & 1 & f_3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1/2(f_1 + 3f_2 - 5f_3) \\ 0 & 1 & 0 & 1/2(f_1 + f_2 - 3f_3) \\ 0 & 0 & 1 & 1/2(f_1 - f_2 + f_3) \end{array}\right).$$

$$(f)\quad \left(\begin{array}{ccccc|c} 1 & 4 & -1 & 4 & 0 & f_1 \\ 3 & 1 & -1 & 2 & 3 & f_2 \\ 0 & 2 & -4 & 3 & -1 & f_3 \end{array}\right) \to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 11/40 & 9/8 & 1/40(-2f_1 + 14f_2 - 3f_3) \\ 0 & 1 & 0 & 17/20 & -1/4 & 1/20(6f_1 - 2f_2 - f_3) \\ 0 & 0 & 1 & -13/40 & 1/8 & 1/40(6f_1 - 2f_2 - 11f_3) \end{array}\right).$$

$$(g) \quad \begin{pmatrix} -2 & 1 & 1 \\ -4 & 2 & 2 \\ 8 & 1 & -1 \\ -7 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$