

## Solutions for Quiz on March 29

We need to diagonalize the matrix,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

First we find the eigenvalues. We need to solve

$$\begin{vmatrix} 1 - \lambda & 0 & 0 \\ -1 & 0 - \lambda & 1 \\ 1 & 1 & 0 - \lambda \end{vmatrix} = 0.$$

We expand the determinant in cofactors of the first row. This row has only one nonzero element, so we have

$$(1 - \lambda)(-1)^{1+1}M_{13} = (1 - \lambda) \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = 0.$$

The minor is given by  $\lambda^2 - 1$ , so the characteristic equation becomes

$$(1 - \lambda)(\lambda^2 - 1) = 0.$$

This has roots  $\lambda_1 = 1$ ,  $\lambda_2 = 1$  and  $\lambda_3 = -1$ . Now we need to find the eigenvectors. We start from the eigenvector corresponding to  $\lambda_3$ . We have,

$$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

From the first equation, we have  $2x_1 = 0$ , which means  $x_1 = 0$ . From the second and third equations, we have,  $0 + x_2 + x_3 = 0$ , which means  $x_2 = -x_3$ . Therefore, the corresponding solution is

$$\alpha \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

**Remark.** This equation can also be solved by row reduction, if you like. We have,

$$\begin{pmatrix} 2 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can see that  $\text{Rank}A = 2$ ,  $m = 3$  (the number of variables), so we have 1 independent variable ( $x_3$ ), and two dependent variables,  $x_1$  and  $x_2$ . The solution given above follows from this immediately.

Next, we find the eigenvectors corresponding to  $\lambda_1$  and  $\lambda_2$ . We have,

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0.$$

Let us solve it by reduction. We have,

$$\begin{pmatrix} 0 & 0 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can see that  $\text{Rank}A = 1$ , and the number of independent variables is  $3 - 1 = 2$ . Independent variables are  $x_1$  and  $x_3$ . We have  $x_1 = -x_2 + x_3$ , so the eigenvector is

$$\beta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \gamma \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Taking any  $\gamma$  and  $\beta$ , we get an eigenvector. We need to have two linearly independent eigenvectors to diagonalize the matrix  $A$ . Let us take  $\beta = 1$  and  $\gamma = 0$ , which gives the eigenvector

$$\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Then, we take  $\beta = 0$ ,  $\gamma = 1$ , which gives the vector

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}.$$

Now, we just need to put the three eigenvectors together to produce the matrix  $P$ ,

$$P = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We have

$$P^{-1}AP = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$