Homework 2- Solutions.

1.

$$f(x) = \frac{1}{x(x^2-x-6)} = \frac{1}{x(x+2)(x-3)} = \frac{A}{x} + \frac{B}{x+2} + \frac{C}{x-3}.$$

Equations for the coefficients:

$$\begin{array}{ll} x^2 & (A+B+C) = 0, \\ x & (-A-3B+2C) = 0, \\ 1 & (-6A) = 1. \end{array}$$

Solve: $A = -1/6$, $B = 1/10$, $C = 1/15$. Have the indefinite integral,

$$\int \left(-\frac{1}{6x} + \frac{1}{10}\frac{1}{x+2} + \frac{1}{15}\frac{1}{x-3} \right) dx =$$

$$\frac{-1}{6} \log|x| + \frac{1}{10} \log|x+2| + \frac{1}{15} \log|x-3|.$$

Next, we evaluate the definite integral.

$$-\frac{1}{6} \log|x| \Big|_1^2 = -\frac{1}{6}(\log 2 - \log 1) = -\frac{1}{6} \log 2.$$

$$\frac{1}{10} \log|x+2| \Big|_1^2 = \frac{1}{10}(\log 4 - \log 3) = \frac{1}{10} \log(4/3).$$

$$\frac{1}{15} \log|x-3| \Big|_1^2 = \frac{1}{15}(\log 1 - \log 2) = -\frac{1}{15} \log 2.$$

All together,

$$-\frac{1}{6} \log 2 + \frac{1}{10} \log(4/3) - \frac{1}{15} \log 2 = -\frac{7}{30} \log 2 + \frac{1}{10} \log(4/3).$$

4 Section 3.2, Problem 32. Write down the right hand side in terms of the Heaviside function,

$$f(t) = 12H(t-4).$$

Calculate Laplace of $f(t)$:

$$\mathcal{L}[f] = \mathcal{L}[12H(t-4)] = 12\mathcal{L}[H(t-4)] = 12\frac{1}{s}e^{-4s},$$

see Example 3.11. Take Laplace of both sides of the equation. Set $\mathcal{L}[y] = Y(s)$.

$$\mathcal{L}[y''] - 2\mathcal{L}[y'] - 3\mathcal{L}[y] = \mathcal{L}[f],$$

$$s^2Y - sy(0) - y'(0) - 2(sY - y(0)) - 3Y = 12\frac{1}{s}e^{-4s},$$

$$Y(s^2 - 2s - 3) - s + 2 = 12\frac{1}{s}e^{-4s},$$

$$Y(s) = \frac{12e^{-4s}}{s^2 - 2s - 3} + \frac{s - 2}{s^2 - 2s - 3}.$$

Take the inverse Laplace, $y(t) = \mathcal{L}^{-1}[Y] =$

$$= 12\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2 - 2s - 3}\right] + \mathcal{L}^{-1}\left[\frac{s - 2}{s^2 - 2s - 3}\right]. \quad (1)$$

First do the first term. From Second Shifting theorem,

$$\mathcal{L}^{-1}[e^{-4s}F(s)] = H(t - 4)f(t - 4).$$

We need to find

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{s^2 - 2s - 3}\right].$$

Partial fractions,

$$\frac{1}{s^2 - 2s - 3} = \frac{A}{s - 3} + \frac{B}{s + 1}.$$

Solve for the coefficients, $A = 1/4$, $B = -1/4$. Therefore,

$$\mathcal{L}^{-1}\left[\frac{1}{s^2 - 2s - 3}\right] = \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t}.$$

Applying the Second Shifting theorem, we obtain the inverse Laplace of the first term of equation (1),

$$12\mathcal{L}^{-1}\left[\frac{e^{-4s}}{s^2 - 2s - 3}\right] = 12H(t - 4)\left(\frac{1}{4}e^{3(t-4)} - \frac{1}{4}e^{-(t-4)}\right) =$$

$$3H(t - 4)\left(e^{3(t-4)} - e^{-(t-4)}\right).$$

The inverse Laplace transform of the second term in equation (1) is calculated by using partial fractions.

5 Section 3.3, Problem 38. Write down the right hand side in terms of the Heaviside function,

$$f(t) = H(t - \pi) \cos(t).$$

In order to take the Laplace transform of this, we would like to use the Second Shifting Theorem. Therefore, we want the argument of the cos to be $(t - \pi)$ instead of t. Recall from trig,

$$\cos(t - \pi) = -\cos(t).$$

Therefore, we have

$$f(t) = H(t - \pi)[-\cos(t - \pi)].$$

Laplace transform,

$$\mathcal{L}[H(t-\pi)(-\cos(t-\pi))] = -\mathcal{L}[H(t-\pi)[\cos(t-\pi)] = -e^{-\pi s}\mathcal{L}[\cos(t)] = -2^{-\pi s}\frac{s}{s^2+1}.$$

Then continue as usual.