MINUTES OF THE MEETING OF 2/22

These notes are devoted to a very careful exposition of a detailed solution of the heat/diffusion equation $u_t = ku_{xx}$ on the interval $[0, \ell]$, $0 \le t < \infty$, with insulation at both ends (Neumann boundary conditions $u_x(0,t) \equiv 0 \equiv u_x(\ell,t)$) and initial heat/material distribution $u(x,0) = \sin\left(\frac{\pi x}{\ell}\right)$, using the method of separation of variables. The notation will be the same that we used in class today.

1. The Separation Step. All attacks of this type on p. d. e.'s begin by seeking solutions of the form $u(x,t) = T(t)X(x)$, where T and X respectively depend only on t and x. In many important cases it is quite natural to classify variables as either "space variables"—in which, generally, solutions are required to satisfy boundary conditions—or (a) "time variable(s)," in which they are required to satisfy initial conditions. For the heat equation the division is obvious and very natural. Since we have $u_t(x,t) = T'(t)X(x)$ and $u_{xx}(x,t) = T(t)X''(x)$ when $u(x,t)$ is of this form, plugging $T(t)X(x)$ into the heat equation will yield

$$T'(t)X(x) = kT(t)X''(x)$$
$$\frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}$$

as the equation to satisfy, and since the second of these equations has equality connecting two functions, one of which depends only on t and the other of which depends only on x, their common value must be a constant.(1) It is customary to give this constant one of the names $\pm\lambda$, where the choice of sign follows various conventions. The one we try to use is

$$\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = -\lambda = (\text{const.}).$$

Thus the problem of finding a function u of this form has turned into two o. d. e. problems: $T' = -\lambda kT$ and $-X'' = \lambda X$. Usually the "space-variable" o. d. e. problem will be a boundary-value problem while the "time-variable" problem will be an initial-value problem, and since boundary-value problems tend to have discrete sequences of solutions that may be hard to find—while one can just simply *solve* initial-value problems—one looks at the boundary-value problem first.

2. The Space-Variable Boundary-Value Problem. Differential equations of the form $-X'' = \lambda X$ are completely studied in beginning o. d. e. courses: one seeks solutions of the form $X = e^{rx}$ and finds out that these exponentials are solutions if and only if $-r^2 = \lambda$, or $r = \pm\sqrt{-\lambda}$, where the roots may be complex. (There are various ways to handle multiple roots, but in this simple case the only possible double root is $r = 0$ which occurs when $\lambda = 0$, and then the solutions have the form $X = A + Bx$.) In all the other cases, one gets solutions that are linear combinations of $e^{\gamma x}$ and $e^{-\gamma x}$ (when $\lambda < 0$ and $-\gamma^2 = \lambda$) or of $e^{i\gamma x}$ and $e^{-i\gamma x}$ (when $\lambda > 0$ and $\gamma^2 = \lambda$), and every solution of the equation must be of that form. For boundary-value work when one space endpoint is $x = 0$ it is usually convenient to regard these solutions in the equivalent forms $A\cosh\gamma x + B\sinh\gamma x$ (when $\lambda < 0$ and $-\gamma^2 = \lambda$) or $A\cos\gamma x + B\sin\gamma x$ (when $\lambda > 0$ and $\gamma^2 = \lambda$), respectively. Note that in both situations we have already divided into the three cases in which λ is negative, zero, or positive, so that when we see γ we have already assumed that it will be positive.

For the Neumann boundary conditions $X'(0) = 0$ and $X'(\ell) = 0$, no (nonzero) hyperbolic-function solutions of the boundary-value problem will exist. The reason is that if $X = A\cosh\gamma x + B\sinh\gamma x$ then $X'(x) = \gamma A\sinh\gamma x + \gamma B\cosh\gamma x$, so $0 = X'(0) = \gamma B$ forces $B = 0$. But now $X' = A\sinh\gamma x$, so the boundary condition at the $x = \ell$ end is $0 = A\sinh\gamma\ell$, and since $\sinh\xi = 0$ holds only for $\xi = 0$, this boundary condition forces $A = 0$ and with it $X \equiv 0$. However, if we try for a linear-function solution (the case $\lambda = 0$) we get something: if $X = A + Bx$ then $X'(x) = B$, so $0 = X'(0) = B$ forces $B = 0$ and thus $X \equiv (\text{const.})$. The condition $X'(\ell) = 0$ then also holds at the $x = \ell$ end, and we have the function $X_0 \equiv 1$ as our first **eigenfunction** of the boundary-value problem $-X'' = \lambda X$: the corresponding **eigenvalue** λ is zero.

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(1) When there is a constant like "k" present in the p. d. e., the solutions usually turn out better-looking if the constant goes with the "time" part of the equation. This is not an inflexible rule.

In contrast to the situation with hyperbolic functions, trig functions furnish an infinite sequence of eigenfunctions of the boundary-value problem. If $X = A\cos\gamma x + B\sin\gamma x$ then $X'(x) = -\gamma A\sin\gamma x + \gamma B\cos\gamma x$, so $0 = X'(0) = \gamma B$ forces $B = 0$ as it did with hyperbolic functions. But now $X' = -\gamma A\sin\gamma x$, so the boundary condition at the $x = \ell$ end is $0 = A\sin\gamma\ell$, and unlike the situation with the hyperbolic sine, the circular $\sin\xi$ is zero whenever $\xi = n\pi$, $n = 1, 2, \ldots$. Thus we can have $A \neq 0$ for any choice of γ that makes $\sin\gamma\ell = 0$, and a complete list of the corresponding λ's is furnished by solving $\gamma\ell = n\pi$ for γ and squaring it to get λ, yielding $\{(n\pi/\ell)^2\}_{n=1}^{\infty}$. So we now have a sequence(2) of eigenfunctions $X_n(x) = \cos\left(\frac{n\pi x}{\ell}\right)$ with corresponding eigenvalues $\lambda_n = (n\pi/\ell)^2$.

3. A Plan for Solving This and More General Problems. We are now in a position to develop a very general game plan for handling initial-value problems of p. d. e.'s in the form of the present one. It reads:

(1) Find the eigenfunctions and eigenvalues of the space boundary-value problem that results when the separation-of-variables technique is applied as it was in §§1–2 above.

(2) Expand the given initial-value function $u(x,t) = \varphi(x)$ in a (finite sum or) series(3)

$$\varphi(x) = \sum_{n=0}^{\infty} A_n X_n(x)\ .$$

(Finding explicit formulas for the coefficients of these series is a question that will be discussed below.)

(3) Using the values of λ_n that came from the space-variable boundary-value problem, find solutions $T_n(t)$ of the time-variable initial-value problems (one for each eigenvalue) that satisfy the "other half of the separated equations;" typically, these solutions will have initial value or initial derivative one and zero respectively, and also vice versa (in the higher order cases). In the present example, the time equation is first-order and for the n-th eigenvalue has the form—and solution with initial value 1

$$\begin{aligned}\frac{T_n'(t)}{kT_n(t)} &= -\lambda_n \\ T_n'(t) &= -\left(\frac{n\pi}{\ell}\right)^2 kT_n(t) \\ T_n(t) &= \exp\left(-\left(\frac{n\pi}{\ell}\right)^2 kt\right)\ .\end{aligned}$$

(4) Insert the solutions of the time initial-value problems (with initial values usually 1) into the corresponding terms of the expansion of the initial-value function $\varphi(x)$ in eigenvalues of the space boundary-value function. The result will be a function

$$u(x,t) = \sum_{n=0}^{\infty} A_n\, T_n(t)\, X_n(x)\ .$$

In the example we're currently working through, it will be(4)

$$u(x,t) = \sum_{n=0}^{\infty} A_n \exp\left(-\left(\frac{n\pi}{\ell}\right)^2 kt\right)\cos\left(\frac{n\pi x}{\ell}\right)\ .$$

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(2) Once we find that it is possible to satisfy the boundary conditions for all A provided that γ is chosen correctly, it suffices to consider only the case in which A=1. The reason is the linearity of the equation $-X''=\lambda X$: we can multiply any function that satisfies it by any constant and get another function that satisfies it. Compare the problem of finding eigenvectors of a matrix: once you have a particular nonzero eigenvector belonging to an eigenvalue λ, any scalar multiple of it will again be an eigenvector belonging to λ.

(3) If the p. d. e. is of higher order in the time variable—the wave equation is our standard example—then both the initial-position function $u(x,t)=\varphi(x)$ and the initial-velocity function $u_t(x,t)=\psi(x)$ will have to be expanded in this manner.

(4) Note that we do not yet have explicit formulas for the coefficients A_n that can be computed from our knowledge of the explicit forms of $\varphi(x)$ and of the eigenfunctions and eigenvalues in this particular example. These things will appear below, however.

> Now consider the functions $u(x,t)$ given by these series (either in the general, rather abstract formulation or in the particular example we're working out). The individual terms of the series satisfy the p. d. e., because each individual term is a multiple of a solution of the p. d. e. that is obtainable by separation of variables, as in §1 above. The p. d. e. is linear, so if the term-by-term differentiations can be justified, then the sum of the series will satisfy the p. d. e. also. When t is set equal to zero, each of the individual "fudge factors" $T_n(t)$ equals 1, so $u(x,0)$ returns to being the series that was supposed to have given $\varphi(x) = \sum_{n=0}^{\infty} A_n X_n(x)$, so the initial value of this solution of the p. d. e. is correct. Voila! we have solved the original p. d. e. problem.

What is amazing about this general game plan is that it actually works, although the deep reasons for that fact do not form a part of a one-semester introductory course in p. d. e. We shall do some of the details, but much of the treatment in this course will depend more on plausible reasoning than on exact proof.

4. Expanding the Initial-Value Function in a Series of Eigenfunctions of the (Space) Boundary-Value Problem. Suppose that we had in fact an expansion

$$\varphi(x) = \sum_{n=0}^{\infty} A_n X_n(x)$$

of a function in a (finite sum or) series of eigenfunctions of a boundary-value problem on an interval $[0,\ell]$ in the (space) variable x. If we happened to know that the eigenfunctions were orthogonal with respect to the usual inner product of functions (or, more generally, a weighted inner product) then we could try the following scheme. Multiply both sides of the expansion by a particular eigenfunction $X_m(x)$, going term-by-term on the series side. Suppose that the result could be integrated term-by-term over $[0,\ell]$ (this would certainly be justifiable in a number of situations). Then one would have the following sequence of results:

$$\begin{aligned} \varphi(x) &= \sum_{n=0}^{\infty} A_n X_n(x) \\ \varphi(x) X_m(x) &= \sum_{n=0}^{\infty} A_n X_n(x) X_m(x) \\ \langle \varphi, X_m \rangle = \int_0^{\ell} \varphi(x) X_m(x)\,dx &= \sum_{n=0}^{\infty} \int_0^{\ell} A_n X_n(x) X_m(x)\,dx = \sum_{n=0}^{\infty} A_n \langle X_n, X_m \rangle\ . \end{aligned} \tag{$*$}$$

But there is only one term that can differ from zero on the r. h. side of ($*$). The reason this is true is that we know

$$\begin{cases} \langle X_n, X_m \rangle = 0 & \text{if } n \neq m; \\ \langle X_n, X_n \rangle > 0 & \text{(assuming } X_n \not\equiv 0\text{—which is how we chose it).} \end{cases}$$

Therefore, there is only one possibly-nonzero term in the series on the extreme r. h. side of ($*$), namely, the one for which the running index-of-summation n equals the index m of the particular eigenfunction with which we started. So ($*$) reduces to

$$\langle \varphi, X_m \rangle = A_m \langle X_m, X_m \rangle$$

$$A_m = \frac{\langle \varphi, X_m \rangle}{\langle X_m, X_m \rangle} = \frac{\int_0^{\ell} \varphi(x) X_m(x)\,dx}{\int_0^{\ell} X_m^2\,dx} \tag{\$}$$

and since the index m was arbitrary, (\$) is a formula from which the numerical values of the coefficients A_n in the series(5) can be computed.

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(5) The indices m and n are dummy indices—it doesn't matter what you call them, as long as you are careful to avoid ambiguity in a single expression. Since m was just "some fixed particular index" in (\$), we can replace it by n everywhere it occurs in that formula.

To see how this works, we return to our illustrative example. Since the Neumann problem for $-X'' = \lambda X$ on a finite interval is a Sturm-Liouville problem, we don't even have to check explicitly that the various eigenfunctions actually are orthogonal—the general Sturm-Liouville machine has done that for us.(6) We do, however, need to know the values of the integrals/inner-products $\langle X_n, X_n\rangle$, and the easiest thing to do is simply to compute them: for $n = 0$

$$\langle X_0, X_0\rangle = \int_0^\ell 1^2\,dx = \ell$$

while for $n \geq 1$ (the integral of the double-angle cosine is zero because it is being integrated over an entire period—or because the sine is zero at both ends of the interval)

$$\langle X_n, X_n\rangle = \int_0^\ell \cos^2\left(\frac{n\pi}{\ell}\right)dx = \int_0^\ell \frac{1+\cos\left(\frac{2n\pi}{\ell}\right)}{2}\,dx = \frac{\ell}{2}\ .$$

The formula ($) for the coefficients thus takes the simple form

$$\begin{aligned} A_0 &= \frac{1}{\ell}\int_0^\ell \varphi(x)\,dx \\ A_n &= \frac{2}{\ell}\int_0^\ell \varphi(x)\cos\left(\frac{n\pi}{\ell}\right)dx \quad \text{for } n \geq 1. \end{aligned}$$

(Having to distinguish two cases bothers some people, who solve the problem by calling the coefficient A_0, which is the constant term $A_0 \cdot 1$ anyway, by the name $\frac{A_0}{2}$. The effect of this is to make the formula we have for $n \geq 1$ above also valid for $n = 0$. This is purely a question of choice of notation. Leaving things as they are above tells you that the value of the constant term is the average value of $\varphi(x)$ for $0 \leq x \leq \ell$, a fact that has nice [quasi-]physical interpretations.)

Now at the very beginning of things we said we would solve the initial-value problem for the heat (or diffusion) equation with the initial values $\varphi(x) = u(x,0) = \sin\left(\frac{\pi x}{\ell}\right)$. In order to do this, we shall actually have to compute the integrals above. This is not much work really: we shall use the same kind of trig identities that turn up in Strauss's §**5.1**. The addition/subtraction formula for the sine reads $\sin(A \pm B) = \sin A\cos B \pm \cos A \sin B$. Subtracting the "$-$" case from the "$+$" case gives

$$\frac{\sin(A+B) - \sin(A-B)}{2} = \cos A \sin B\ .$$

The integrands in the inner products $\langle \varphi, X_n\rangle = \int_0^\ell \sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)dx$ can therefore be rewritten in the form

$$\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right) = \frac{\sin\left(\frac{(n+1)\pi x}{\ell}\right) - \sin\left(\frac{(n-1)\pi x}{\ell}\right)}{2} \tag{\#}$$

and integrating the r. h. side is elementary. There is a small problem, in that for $n = 1$ the second sine is identically zero, and uncritical formal integration of that term might cause trouble; so let us compute the inner products for $n = 0$ and $n = 1$ separately before going on to the case $n \geq 2$, in which we can simply integrate formally and not worry about division-by-zero errors.

$$\begin{aligned} \langle \varphi, 1\rangle &= \int_0^\ell \sin\left(\frac{\pi x}{\ell}\right)dx = \frac{\ell}{\pi}\cdot[-\cos\pi + \cos 0] = \frac{2\ell}{\pi} \\ A_0 &= \frac{1}{\ell}\frac{2\ell}{\pi} = \frac{2}{\pi}\ ; \\ \langle \varphi, X_1\rangle &= \int_0^\ell \sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi x}{\ell}\right)dx = \frac{1}{2}\frac{\ell}{\pi}\sin^2\left(\frac{\pi x}{\ell}\right)\Big]_0^\ell = 0 - 0 = 0 \\ A_1 &= 0\ . \end{aligned}$$

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(6) Though if it really makes you feel better to see the integrals computed by patient unrewarding toil, see Strauss, pp. 101–104.

For $n \geq 2$, we can simply integrate (#) above: using the fact that $\cos m\pi = (-1)^m$ and that its value depends only on the parity of m—it's $+1$ if m is even and -1 if m is odd—we have

$$\begin{aligned}\langle \varphi, X_n \rangle = \int_0^\ell \sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)dx &= \frac{1}{2}\int_0^\ell \left\{\sin\left(\frac{(n+1)\pi x}{\ell}\right) - \sin\left(\frac{(n-1)\pi x}{\ell}\right)\right\}dx \\ &= \frac{1}{2}\left[-\frac{\ell}{(n+1)\pi}\cos\left(\frac{(n+1)\pi x}{\ell}\right) + \frac{\ell}{(n-1)\pi}\cos\left(\frac{(n-1)\pi x}{\ell}\right)\right]_0^\ell \\ &= \frac{1}{2}\left\{-\frac{\ell}{(n+1)\pi}\left[(-1)^{n+1}-1\right] + \frac{\ell}{(n-1)\pi}\left[(-1)^{n-1}-1\right]\right\} .\end{aligned}$$

We can now begin to effect some simplifications. First of all, if n is odd then $n+1$ and $n-1$ will be even, so the quantities $\left[(-1)^{n+1}-1\right]$ will equal $1-1=0$ in that case. Hence all the odd-indexed A_n's are zero. On the other hand, when n is even $(-1)^{n\pm1} = -1$, so the so the quantities $\left[(-1)^{n+1}-1\right]$ will equal $-1-1=-2$ in that case. Thus for even n the last set-off expression in braces above can be condensed into

$$-\frac{\ell}{(n+1)\pi}\left[(-1)^{n+1}-1\right] + \frac{\ell}{(n-1)\pi}\left[(-1)^{n-1}-1\right] = \frac{2\ell}{\pi}\left\{\frac{1}{n+1} - \frac{1}{n-1}\right\} = \frac{-4\ell}{\pi}\,\frac{1}{n^2-1} ,$$

and thus $A_n = \dfrac{2}{\ell}\langle \varphi, X_n \rangle = \dfrac{-4}{\pi}\,\dfrac{1}{n^2-1}$ for even $n \geq 2$, $A_n = 0$ for odd $n \geq 1$. The generic "even n" can be written as $n = 2j$ where j is a whole number, so the expansion

$$\varphi(x) = \sum_{n=0}^{\infty} A_n X_n(x)$$

reduces to

$$\sin\left(\frac{\pi x}{\ell}\right) = \varphi(x) = A_0 + \sum_{j=1}^{\infty} A_{2j}X_{2j}(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{4j^2-1}\cos\left(\frac{2j\pi x}{\ell}\right) .$$

5. Inserting the Time-Dependence Factors. In (3) on p. 2 above, we had already computed the solutions of the initial-value problems that determined the $T_n(t)$ corresponding to each eigenvalue λ_n and having initial value 1: it was $\exp\left(-\left(\dfrac{n\pi}{\ell}\right)^2 kt\right)$. All that was missing in the expansion of $u(x,t)$ at the bottom of that same p. 2 was explicit values for the coefficients A_n, and we now have those: the odd-indexed ones are zero, and the even-indexed ones were computed in §4 above. We must take a little care and pay attention to the fact that the index n for the eigenvalues/functions is equal to $2j$ in the series in which $\varphi(x)$ was expanded, so the n in the functions $T_n(t)$ must also be replaced by $n = 2j$ in each of these factors. The solution thus takes the final form

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{4j^2-1}\exp\left(-\left(\frac{2j\pi}{\ell}\right)^2 kt\right)\cos\left(\frac{2j\pi x}{\ell}\right) .$$

You can't see the factor $T_0(t)$ in the index-zero term $\dfrac{2}{\pi}T_0(t)$, because $\lambda_0 = 0$ and therefore $T_0(t) \equiv 1$.

6. Things Deducible from the Form of the Series Solution. Examination of the series solution just obtained for this particular equation illustrates some interesting things—some true in general, some rather peculiar to the initial-value function $\varphi(x)$ in this particular problem.

(1) The series expansion of this particular initial value function

$$\sin\left(\frac{\pi x}{\ell}\right) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty}\frac{1}{4j^2-1}\cos\left(\frac{2j\pi x}{\ell}\right) \qquad 0 \leq x \leq \ell$$

converges for all values of x: because $|\cos\xi| \le 1$ for all real values of x, each term of the series is no larger in absolute value than its coefficient A_{2j}, which is a constant multiple of $\dfrac{1}{4j^2-1}$. The integral test of freshman calculus says that the series $\displaystyle\sum_{j=1}^{\infty} \frac{1}{4j^2-1}$ converges if and only if the integral $\displaystyle\int_1^{\infty} \frac{dx}{4x^2-1} < +\infty$, and this integral is indeed finite. Thus the series expansion converges absolutely; moreover, because the terms can be estimated by a convergent series of constants, the series converges uniformly to its limit function, which must be a continuous function.(7) (We are still taking for granted that this series, and [more generally] all series constructed in this way, do in fact converge for $0 \le x \le \ell$ to the function $\varphi(x)$ from which the coefficients were computed. The details of proving convergence—and legitimizing the method we just used—occupy Strauss's §§**5.4**–**5.5**; but sometimes funny things happen at the endpoints $x = 0$ and $x = \ell$, as you can see by plugging $x = 0$ and $x = \ell$ into the series for the identically-1 function that Strauss gives you in his §**4.1 #2** problem on p. 87.)

(2) The series for the solution function,

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{4j^2-1}\, \exp\left(-\left(\frac{2j\pi}{\ell}\right)^2 kt\right) \cos\left(\frac{2j\pi x}{\ell}\right)\ ,$$

however, converges *very rapidly* for all values of $t > 0$. The factors $\exp\left(-\left(\frac{n\pi}{\ell}\right)^2 kt\right)$ fall off with increasing n like $e^{-(\text{const.})n^2}$, *much faster* than "geometrically fast." Thus even for fairly small values of $t > 0$, only a few terms of the series are necessary in order to get a very good approximation to $u(x,t)$, and in fact the approximation becomes better and better as $t \to \infty$. These properties are preserved even after term-by-term differentiation of the series, because differentiations multiply the n-th term by a power of n, which grows only polynomially as $n \to \infty$, while $\exp\left(-\left(\frac{n\pi}{\ell}\right)^2 kt\right)$ is decaying at negative-exponential-squared speed. You might test this yourself: use L'Hôpital's rule to find the limit (for example) of $n^4\, e^{-n^2}$.

(3) The Neumann problem for the heat/diffusion equation gives insulated ends to the heated rod or closed ends to the tube with diffusing dye in it, respectively. Physical intuition says that after a long time, the temperature throughout the rod will be very near the space average of the initial temperature, or the concentration of the dye will be very near the space average of the original concentration (the given quantity of dye will be uniformly distributed throughout the tube). The explicit series solution

$$u(x,t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{4j^2-1}\, \exp\left(-\left(\frac{2j\pi}{\ell}\right)^2 kt\right) \cos\left(\frac{2j\pi x}{\ell}\right)$$

tells you how this happens, explicitly and quantitatively. The $\dfrac{2}{\pi}$ constant term was actually computed as the space average of $\varphi(x)$ over $[0,\ell]$. The terms of the series go to zero individually as $t \to \infty$, but in fact we can make a very easy estimate of the sum of the entire series: the "fudge factor" T_2 is the largest of the fudge factors and all the cosines are bounded by 1, so the difference between $u(x,t)$ and the constant term in the series can be estimated by a single exponentially-decaying function of t:

$$\left|\frac{4}{\pi}\sum_{j=1}^{\infty} \frac{1}{4j^2-1}\, \exp\left(-\left(\frac{2j\pi}{\ell}\right)^2 kt\right) \cos\left(\frac{2j\pi x}{\ell}\right)\right| \le \frac{4}{\pi}\cdot \exp\left(-\left(\frac{2\pi}{\ell}\right)^2 kt\right)\cdot \sum_{j=1}^{\infty} \frac{1}{4j^2-1}\ .$$

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(7) People who have had an advanced-calculus course may recognize here the Weierstraß M-test for uniform convergence of a series.