

LEIBNIZ'S FORMULA FOR DIFFERENTIATING INTEGRALS THAT DEPEND ON PARAMETERS

1. Leibniz's Formula. Consider functions that are defined by integrals whose limits and/or integrand depend on a parameter, *i.e.*, functions of the form

$$F(t) = \int_{a(t)}^{b(t)} f(x, t) dx .$$

One may need $\frac{dF}{dt}$: how does one find it? The easiest way to see what happens in general is to consider the function of three variables

$$\varphi(a, b, t) = \int_a^b f(x, t) dx ,$$

realize that $F(t) = \varphi(a(t), b(t), t)$, and apply the chain rule to give

$$\frac{dF}{dt} = \frac{\partial \varphi}{\partial a} \cdot \frac{da}{dt} + \frac{\partial \varphi}{\partial b} \cdot \frac{db}{dt} + \frac{\partial \varphi}{\partial t} \cdot \frac{dt}{dt} ,$$

thus breaking the hard problem into three easier ones. Finding the partial derivatives with respect to a and b offers no problems, provided we assume that the integrand $f(x, t)$ is a continuous function of x for each value of t , since under that assumption the fundamental theorem of the integral calculus gives

$$\frac{\partial}{\partial b} \varphi(a, b, t) = f(b, t)$$

and similarly, since " $\int_a^b = -\int_b^a$,"

$$\frac{\partial}{\partial a} \varphi(a, b, t) = -f(a, t) .$$

For $\frac{\partial \varphi}{\partial t}$ we have to work harder, and go back to the definition of the derivative to see what the value might be. From the definition of the derivative we get (treating a and b as constant)

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \lim_{h \rightarrow 0} \frac{\varphi(a, b, t+h) - \varphi(a, b, t)}{h} = \lim_{h \rightarrow 0} \frac{1}{h} \left[\int_a^b f(x, t+h) dx - \int_a^b f(x, t) dx \right] \\ &= \lim_{h \rightarrow 0} \int_a^b \frac{f(x, t+h) - f(x, t)}{h} dx \stackrel{?}{=} \int_a^b \lim_{h \rightarrow 0} \frac{f(x, t+h) - f(x, t)}{h} dx = \int_a^b \frac{\partial f(x, t)}{\partial t} dx . \end{aligned}$$

Assuming that the equal-sign with a "?" over it is valid and making other assumptions to insure that the chain rule holds in this situation, we would then have

$$\frac{dF}{dt} = \frac{\partial \varphi}{\partial a} \cdot \frac{da}{dt} + \frac{\partial \varphi}{\partial b} \cdot \frac{db}{dt} + \frac{\partial \varphi}{\partial t} \cdot \frac{dt}{dt} = -f(a, t) \frac{da}{dt} + f(b, t) \frac{db}{dt} + \int_a^b \frac{\partial f(x, t)}{\partial t} dx$$

(remember, $\frac{dt}{dt} = 1$), which is **Leibniz's Formula**.

What do we need to know in order to make this formal computation rigorous? For the chain rule, we need the partial derivatives of $\varphi(a, b, t)$ with respect to its variables to be continuous functions of (a, b, t) . For the partials with respect to a and b there is little problem: if we assume that $f(x, t)$ is continuous as a function of (x, t) , then the partials $-f(a, t)$ and $f(b, t)$ depend continuously on (a, b, t) . For the partial with respect to t , an easily verified condition would be that $\frac{\partial f}{\partial t}$ is a continuous function of (x, t) ; it is tedious but not difficult to show that

$$(a, b, t) \mapsto \int_a^b g(x, t) dx$$

is a continuous function of (a, b, t) whenever $g(x, t)$ is a continuous function of (x, t) .

We still have to justify “taking the limit under the integral sign” when we compute $\frac{\partial \varphi}{\partial t}$. If we make the assumption that $\frac{\partial f}{\partial t}$ is a continuous function of (x, t) , we can do this in the following way. The mean-value theorem of the differential calculus tells us that for each value of x between a and b there is a value of t between t and $t + h$, call it τ_x , for which the difference quotients indicated below equal the derivative indicated below:

$$\frac{f(x, t + h) - f(x, t)}{h} = \frac{\partial f(x, \tau_x)}{\partial t}.$$

It follows that

$$\frac{f(x, t + h) - f(x, t)}{h} - \frac{\partial f(x, t)}{\partial t} = \frac{\partial f(x, \tau_x)}{\partial t} - \frac{\partial f(x, t)}{\partial t}.$$

Now t and τ_x are very close to each other—within $|h|$ —so by the (uniform with respect to both variables) continuity of $f(x, t)$ we can find a $\delta > 0$ such that if $|h| < \delta$, so that also $|\tau_x - t| < \delta$, then⁽¹⁾

$$\left| \frac{f(x, t + h) - f(x, t)}{h} - \frac{\partial f(x, t)}{\partial t} \right| = \left| \frac{\partial f(x, \tau_x)}{\partial t} - \frac{\partial f(x, t)}{\partial t} \right| \leq \frac{\epsilon}{|b - a|},$$

where $\epsilon > 0$ is any preassigned small number. Integrating this estimate gives

$$\left| \int_a^b \frac{f(x, t + h) - f(x, t)}{h} dx - \int_a^b \frac{\partial f(x, t)}{\partial t} dx \right| \leq \int_a^b \left| \frac{\partial f(x, \tau_x)}{\partial t} - \frac{\partial f(x, t)}{\partial t} \right| dx \leq \frac{\epsilon}{|b - a|} \cdot |b - a| = \epsilon.$$

By the “ ϵ - δ ” definition of limit, this proves that

$$\lim_{h \rightarrow 0} \int_a^b \frac{f(x, t + h) - f(x, t)}{h} dx = \int_a^b \frac{\partial f(x, t)}{\partial t} dx.$$

2. Our Computational Example. Just to see that this really works, I suggested that people in the class try computing the derivative of

$$F(t) = \int_t^{t^2} \cos(tx) dx$$

both by computing the (elementary) integral directly and differentiating the result and by applying Leibniz’s formula. The integral is

$$F(t) = \int_t^{t^2} \cos(tx) dx = \left. \frac{\sin(tx)}{t} \right]_{x=t}^{x=t^2} = \frac{\sin(t^3) - \sin(t^2)}{t}$$

and its derivative, by the quotient rule, is

$$\frac{dF}{dt} = \frac{t \cdot [3t^2 \cos(t^3) - 2t \sin(t^2)] - [\sin(t^3) - \sin(t^2)] \cdot 1}{t^2}.$$

On the other hand, applying Leibniz’s formula gives

$$\begin{aligned} F'(t) &= -1 \cdot \cos(t^2) + 2t \cdot \cos(t^3) - \int_t^{t^2} x \sin(tx) dx \\ &= -\cos(t^2) + 2t \cdot \cos(t^3) - \left[-\frac{\sin(t^3) + t^3 \cos(t^3) + \sin(t^2) - t^2 \cos(t^2)}{t^2} \right] \end{aligned}$$

⁽¹⁾ The astute observer will notice that I am assuming $a \neq b$ here. However, if the two endpoints were equal, then the integral would be zero and one would not have to estimate it!

and it is pure ugly high-school algebra (which any sensible person would do in **Maple** or **MATLAB**⁽²⁾) to put this second expression over the common denominator t^2 , obtaining the same result that we had by explicit integration of the integral that defined $F(t)$.

So why bother with Leibniz's formula? Because if the definition of $F(t)$ had (for example) been changed to

$$F(t) = \int_t^{t^2} \cos(tx^2) dx$$

the explicit integration would have become impossible. However, we would still have had a formula

$$F'(t) = -1 \cdot \cos(t^3) + 2t \cdot \cos(t^5) - \int_t^{t^2} x^2 \sin(tx^2) dx$$

for the derivative of $F(t)$, and we might be able to do something with that, *e.g.*, make estimates of its size, or write a power series for it, etc.

⁽²⁾ Indeed, a sensible person would have a symbolic-manipulation program do the integration-by-parts, as I did.