THE 2-DIMENSIONAL LAPLACIAN IN POLAR COÖRDINATES

Recall that with $x = r\cos\theta$ and $y = r\sin\theta$ as usual, the chain rule (in vector-matrix form) gives

$$\begin{bmatrix} \dfrac{\partial f}{\partial r} \\ \dfrac{\partial f}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} \begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix}$$

and therefore (where we invert the matrix of coefficients by the usual determinant method)

$$\begin{bmatrix} \dfrac{\partial f}{\partial x} \\ \dfrac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & \dfrac{-\sin\theta}{r} \\ \sin\theta & \dfrac{\cos\theta}{r} \end{bmatrix} \begin{bmatrix} \dfrac{\partial f}{\partial r} \\ \dfrac{\partial f}{\partial \theta} \end{bmatrix} .$$

The first row of this product can be written in operator form as

$$\frac{\partial}{\partial x} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$

and so applying it twice to a function f gives

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \cos\theta \frac{\partial}{\partial r}\left\{\cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right\} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\left\{\cos\theta \frac{\partial f}{\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial \theta}\right\} \\ &= \cos\theta\left[\cos\theta\frac{\partial^2 f}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial f}{\partial \theta} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta}\right] - \frac{\sin\theta}{r}\left[-\sin\theta\frac{\partial f}{\partial r} + \cos\theta\frac{\partial^2 f}{\partial\theta\partial r} - \frac{\cos\theta}{r}\frac{\partial f}{\partial\theta} - \frac{\sin\theta}{r}\frac{\partial^2 f}{\partial\theta^2}\right] \\ &= \cos^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{\sin^2\theta}{r}\frac{\partial f}{\partial r} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2} . \qquad (f_{xx}) \end{aligned}$$

Similarly, the second row of that matrix product can be written in operator form as

$$\frac{\partial}{\partial y} = \sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}$$

and so applying it twice to a function f gives

$$\begin{aligned} \frac{\partial^2 f}{\partial y^2} &= \sin\theta \frac{\partial}{\partial r}\left\{\sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right\} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\left\{\sin\theta \frac{\partial f}{\partial r} + \frac{\cos\theta}{r}\frac{\partial f}{\partial \theta}\right\} \\ &= \sin\theta\left[\sin\theta\frac{\partial^2 f}{\partial r^2} - \frac{\cos\theta}{r^2}\frac{\partial f}{\partial \theta} + \frac{\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta}\right] + \frac{\cos\theta}{r}\left[\cos\theta\frac{\partial f}{\partial r} + \sin\theta\frac{\partial^2 f}{\partial\theta\partial r} - \frac{\sin\theta}{r}\frac{\partial f}{\partial\theta} + \frac{\cos\theta}{r}\frac{\partial^2 f}{\partial\theta^2}\right] \\ &= \sin^2\theta\frac{\partial^2 f}{\partial r^2} + \frac{\cos^2\theta}{r}\frac{\partial f}{\partial r} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 f}{\partial r\partial\theta} - \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial f}{\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 f}{\partial\theta^2} . \qquad (f_{yy}) \end{aligned}$$

The functions $\dfrac{\partial f}{\partial\theta}$ and $\dfrac{\partial^2 f}{\partial r\partial\theta}$ on the lines (f_{xx}) and (f_{yy}) have coefficients that are equal but opposite in sign, so when the two lines are added these partial derivatives disappear. On the other hand, all the remaining partial derivatives occur once with the coefficient $\cos^2\theta$ and once with the coefficient $\sin^2\theta$. Therefore, when the two lines are added the surviving terms have the form

$$\begin{aligned} \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= (\cos^2\theta + \sin^2\theta)\cdot\left[\frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}\right] \\ &= \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2} \end{aligned}$$

which is the polar-coördinate form of the Laplacian in dimension 2.