MATH 423:01 FALL 2001 — PROBLEM SHEET 3

For The Rules, see problem sheet 1. Your solutions for the problems on this sheet are due in class on Tuesday November 6^{th}, or delivered to the Mathematics Undergraduate Office, Hill Center 303 (Busch), no later than 4PM on Wednesday November 7^{th}. Do **NOT** slide them under the instructor's office door; hand them to a human or presume them lost.

Throughout this problem sheet you are encouraged to use `Maple`, `Mathematica`, `Matlab` or at least a table of integrals when you have some particular function to integrate. There are enough ways to screw up conceptually without having to make mistakes in (for example) iterated integrations by parts.

1. This problem is a small reminder of Leibniz's rule, combined with an amusing application of Fourier series. The "vibrating spring" or "simple harmonic oscillator" with external force function or "driving function" $f(t)$, and position zero and velocity zero at time zero, is governed by the o. d. e. initial-value problem

$$mx'' + kx = f(t) \qquad x(0) = 0, \;\; x'(0) = 0.$$

The convolution-integral or "impulse-response" solution of this initial-value problem, found (for example) by formal computation with the Laplace transform, is given by

$$x(t) = \frac{1}{m\omega_0} \int_0^t \sin(\omega_0(t-\tau)) f(\tau)\, d\tau \quad \text{for } t \geq 0 \tag{1}$$

where $\omega_0 = \sqrt{k/m}$ is the natural angular velocity of the oscillator.

1. (a): **Use Leibniz's rule** to verify that the function $x(t)$ defined in (1) does in fact satisfy the o. d. e. that it is supposed to satisfy, and does take the correct initial values.(1)

1. (b): Suppose that the function $f(t)$ is periodic with period 2ℓ, and is represented by the Fourier sine series

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{\ell}\right) \,,$$

whose coefficients would be given by $b_n = \dfrac{1}{\ell}\displaystyle\int_{-\ell}^{\ell} f(\theta)\sin\left(\frac{n\pi\theta}{\ell}\right) d\theta$. **Compute the result of applying** (1) **formally term-by-term** to the Fourier series for $f(t)$. Then **compute the coefficients** in the (full) Fourier series for $x(t)$ **from their integral expressions**: *e.g.*, the sine coefficients would be given by

$$B_n = \frac{1}{\ell}\int_{-\ell}^{\ell} x(\theta)\sin\left(\frac{n\pi\theta}{\ell}\right) d\theta$$

where the function $x(t)$ is given by the integral (1). Do you get any surprises?

1. (c): **Discuss** anything that seems to violate a "general rule" in **(b)** above when certain relations hold among n, ℓ and ω_0.(2)

2. (a): Consider the function $P_1(x)$ defined to equal $x - \dfrac{1}{2}$ on the interval $(0,1)$, to equal zero at $x = 0$ and $x = 1$, and extended to be odd and periodic with period $\ell = 1$ on the whole real line. **Show that its Fourier (sine) expansion is**

$$P_1(x) = -\sum_{j=1}^{\infty} \frac{\sin 2\pi j x}{\pi j} \,.$$

This function has the rather long name of *the periodic extension of the first Bernoulli polynomial*, the first Bernoulli polynomial being $B_1(x) = x - \dfrac{1}{2}$.

(1) Yes, it is possible to do this by using the addition formula for the sine, but use Leibniz's rule instead.

(2) This question is left vague for the purpose of not furnishing too much information. If you don't at first see what's going on here, consider the function $x(t)$ that arises in the case $f(t)=\sin(\omega_0 t)$. (You may use the integral to compute it, or solve the initial-value problem by some other means at your disposal, for example, the Laplace transform method or a method of undetermined coefficients.)

2. (b): More generally, consider the functions

$$P_{2k}(x) = (-1)^{k+1} \sum_{j=1}^{\infty} \frac{2\cos 2\pi j x}{(2\pi j)^{2k}} \quad \text{(definition for even indices)}$$

$$P_{2k+1}(x) = (-1)^{k+1} \sum_{j=1}^{\infty} \frac{2\sin 2\pi j x}{(2\pi j)^{2k+1}} \quad \text{(definition for odd indices).}$$

For $2k$ or $2k+1$ larger than 1, these series converge to continuous functions by comparison with a series $\sum 1/n^p$ where $p = 2k$ or $2k+1$. **Use the result** of Strauss's problem §**5.4 #11** (pp. 130–131) to show that each $P_r(x)$ is an antiderivative (= integral) of the previous $P_{r-1}(x)$, with the constant of integration determined by the requirement that $\int_0^1 P_r(x)\,dx = 0$. This implies, for example, that $P_2(x) = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$ (where the constant-of-integration $1/12$ had to be added to make the integral over $[0,1]$ come out zero). **Sum the series** for $P_3(x)$ and $P_4(x)$ similarly (for $0 \le x \le 1$, of course; in each case, the series will represent periodic extensions of the polynomials that you get).

2. (c): Plug in a suitable value (or suitable values) of x in $P_2(x)$ and $P_4(x)$ respectively to **sum the series**

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{1}{n^4}$$

explicitly (giving explicit numerical values in terms of rational numbers and a well-known constant). You will need your results from **(c)** above.

3. This problem[(3)] combines elements of the second and third problems of the previous set; you might want to delay working on it until the suggested solutions of that set have been posted (this should occur no later than Sunday, Oct. 27^{th}).

3. (a): **Consider** the eigenvalue problem on $[0,\ell] = [0,1]$ given by

$$X'' = -\lambda X \quad \text{for } x \in [0,1]$$
$$X(0) = 0 \quad \text{(Dirichlet condition at } x = 0\text{)}$$
$$X'(1) = 2 \cdot X(1) \quad \text{(Robin condition at } x = 1\text{, with } a_\ell = -2\text{).}$$

Use `Maple`, `MATLAB`, or software of your own choice or devising to **find the single negative eigenvalue** of this problem (with corresponding eigenfunction $X_0(x) = \sinh \gamma_0 x$, where $\lambda_0 = -\gamma_0^2$) **and a positive eigenvalue** of your choice (with corresponding eigenfunction $X_k(x) = \sin \beta_k x$, where $\lambda_k = \beta_k^2$). Avoid making an eccentric choice: numbers that are too large will corrupt your results in the next part of this problem.

3. (b): **Use the numerical integration** routine in your software package (or write a routine) to compute the inner product

$$\langle X_0, X_k \rangle = \int_0^1 \sinh \gamma_0 x \sin \beta_k x \, dx \, .$$

Discuss the correspondence of your numerical result with what the general theory leads you to expect.

4. Read, carefully, the posted solutions of Strauss's problems §**5.4 #19** and **5.4 #20** (p. 131). Then **modify the technique** illustrated in the solution of **5.4 #20** to **evaluate the normalization constants** for problem 3 of this semester's Problem Sheet 2. Correct results will be functions of $\sqrt{\lambda}$, because for Robin conditions the normalization constants don't all equal 1. {Note that the general form for the eigenfunctions of that problem was $\frac{\sin(x\sqrt{\lambda})}{\sqrt{\lambda}}$, so the calculations will be considerably simpler than they were in Strauss's §**5.4 #20**.} Try to write your results in as civilized-looking a form as possible.

(3) The genesis of this problem was a question raised in class.