MATH 423:01 FALL 2001 — FINAL PROBLEM SHEET

Please submit solutions for these problems on or before the scheduled examination time/date: 2000 EST (= 8:00 PM) Tuesday, December 18th, 2001 or (worst case) drop them off in the main Mathematics Department office, Hill 303 (Busch) before 12:00 noon Thursday, December 20th, 2001.

1. Find the solution $u(x, y)$ of the first-order linear homogeneous p. d. e.

$$\frac{\partial u}{\partial x} - 3\,\frac{\partial u}{\partial y} = 0$$

that satisfies the condition $u(x, x) = x^2$ for all real x. Be careful to understand the problem before offering an uncritical solution!

2. Suppose a function $f(x)$ defined on the interval $0 \le x \le \pi$, and sufficiently "nice" that its formal Fourier sine series converges to it, is symmetric with respect to the line $x = \pi/2$, that is, satisfies the identity $f(x) = f(\pi - x)$.

2. (a) Use the symmetry property to show that the Fourier sine series of $f(x)$ on $[0, \pi]$ has only odd-indexed terms, that is, its Fourier sine coefficients defined by

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

satisfy $B_n = 0$ if n is even ($n = 2j$), and furthermore that the odd-indexed coefficients B_{2j-1} can be calculated as[(1)]

$$B_{2j-1} = \frac{4}{\pi} \int_0^{\pi/2} f(x) \sin((2j-1)x) \, dx, \qquad j = 1, 2, \ldots .$$

2. (b) Use the result of **(a)** (even if you have not completely proved it) to find a Fourier series of the stated form that converges to $f(x) = x^2$ for $0 \le x \le \pi/2$ and to its reflection in the line $x = \pi/2$ for $\pi/2 \le x \le \pi$. Sketch the graph of the periodic function to which this series converges.

3. (a) Use the method of separation of variables (in rectangular coördinates) to find the harmonic function in the semiïnfinite plane strip $\{(x, y) : 0 \le x \le \pi,\ 0 \le y\}$ that satisfies the "boundary conditions"

$$u(0, y) \equiv 0 \equiv u(\pi, y), \qquad u(x, 0) = h(x), \qquad \lim_{y \to +\infty} u(x, y) = 0.$$

Briefly explain how the the maximum principle offered in §3 of the notes `cauchyk.pdf` can be adapted to this situation, using the condition $\lim\limits_{y \to +\infty} u(x, y) = 0$. You need not go into excruciatingly fine detail.

4. Use the first Green's formula [formula (G1) of Strauss, p. 171] and a suitable modification of the energy method that Strauss uses to prove **UNIQUENESS OF DIRICHLET'S PROBLEM** on p. 173 to prove the **uniqueness of the Robin problem**: if $D \subseteq \mathbb{R}^3$ is a (bounded) domain (with piecewise-smooth boundary for which the divergence theorem is valid) and $a > 0$ is a constant, then the only solution of $\nabla^2 u = 0$ in D satisfying $\dfrac{\partial u}{\partial n} + au = 0$ on the boundary of D is the identically-zero solution. It follows that the solution of the boundary-value problem $\nabla^2 u = 0$ in D, $\dfrac{\partial u}{\partial n} + au = \phi$ on the boundary of D is uniquely determined.

5. (a) Show that if $w(u, v)$ is a harmonic function ($w_{uu} + w_{vv} = 0$), then so is the function defined by $W(x, y) = w(x^2 - y^2, 2xy)$. {Hint: Two approaches that are relatively painless are (i) use polar coördinates; (ii) if you know some complex function theory, use it. Less painless is (iii): compute out the Laplacian w. r. t. x, y hammer-and-tongs in rectangular coördinates, although it is possible that `Maple` could be coaxed into doing this.}

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(1) Note that this formula can only be used for odd indices: used uncritically for all "n" and not exclusively for $n{=}2j{-}1$, it can give nonsensical results.

5. (b) Show that the mapping from the first quadrant $\{(x, y) : x > 0,\ y > 0\}$ to the upper half-plane $\{(u, v) : v > 0\}$ given by

$$\begin{aligned} u &= x^2 - y^2 \\ v &= 2xy \end{aligned}$$

is onto (*i.e.*, every point in the upper half-plane is the image of some point in the first quadrant) and 1-1 (*i.e.*, that point in the first quadrant is uniquely determined). {Hint: A polar-coördinate argument is easy to give: you need the double-angle formulas. A purely algebraic proof using the quadratic formula can also be given.}

5. (c) [more difficult] Use the Poisson/Cauchy kernel that solves the Dirichlet problem for $\nabla^2 u = 0$ in the upper half-plane, together with **(a)** and **(b)** above (you may assume these even if you haven't proved them) to give an integral formula for solving the Dirichlet problem for the first quadrant (with boundary data given on the nonnegative x-axis and the nonnegative y-axis). The things you need about the Cauchy kernel, which is the Poisson kernel for the upper half-plane, can be found in §§1–2 of the file `cauchyk.pdf`, rewritten last September and accessible from the course web page.