STURM-LIOUVILLE OPERATORS I: ORTHOGONALITY

0. Introduction. The usual elementary presentation of the orthogonality relations—the formulas numbered (2), (3), (4), (5), (7), (9), (10), etc., in Strauss's Chapter 4—for the eigenfunctions of $-X'' = \lambda X$ (with any of the three kinds of boundary conditions that we have been considering) may seem to come from some species of fortunate miracle. The trigonometric functions just happen to satisfy just the right identities. There is an alternative point of view that perceives the reason for the orthogonality relations as something like the fact that eigenvectors of a symmetric matrix that belong to different eigenvalues are always orthogonal. The differential operator is something like a symmetric matrix—though the symmetry actually comes from the combination of the differential operator and the boundary conditions—and something like an inner product is present. While we shall only present heuristic analogies below, these considerations can be (and are) made quite rigorous and explicit when the relevant analytical tools are available. In this first course we shall prove what is easily proved, leaving untouched some harder questions.

1. Inner Products of Vectors, and Self-Adjointness. Everybody knows the definition of the usual inner (or dot) product of two vectors in $\mathbb{R}^n$: if $\mathbf{a} = (a_1, \ldots, a_n)^T$ and $\mathbf{b} = (b_1, \ldots, b_n)^T$, then(1)

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^{n} a_j \, b_j = \mathbf{b}^T \mathbf{a} \, .$$

If one wants to work with complex scalars, then the inner product of $\mathbf{a} = (a_1, \ldots, a_n)^T \in \mathbb{C}^n$ and $\mathbf{b} = (b_1, \ldots, b_n)^T \in \mathbb{C}^n$ is

$$\langle \mathbf{a}, \mathbf{b} \rangle = \sum_{j=1}^{n} a_j \, \overline{b}_j = \mathbf{b}^H \mathbf{a} \, ,$$

where the overbar indicates complex conjugation(2) and the superscript ^{H} indicates the conjugate transpose: A^H is then called the **adjoint** of A. In both cases, the length-squared of a vector $\mathbf{a}$ is $\langle \mathbf{a}, \mathbf{a} \rangle = \|\mathbf{a}\|^2$. A matrix A (usually with real entries) is called **symmetric** if $A = A^T$ and a matrix (with entries that may be complex) is called **Hermitean** or **self-adjoint** if $A = A^H$. Evidently if a matrix A has real entries and is symmetric, it is also self-adjoint when one thinks of its entries as being complex numbers (whose imaginary parts just happen to equal zero). Both $A \mapsto A^T$ and $A \mapsto A^H$ reverse the order of matrix multiplication, *i.e.*, $(AB)^T = B^T A^T$ and $(AB)^H = B^H A^H$. The interest in these matrices is that the transpose or adjoint "passes across the inner product": if the scalars are $\mathbb{R}$ then for any matrix A

$$\langle A\mathbf{a}, \mathbf{b} \rangle = \mathbf{b}^T A\mathbf{a} = (A^T \mathbf{b})^T \mathbf{a} = \langle \mathbf{a}, A^T \mathbf{b} \rangle \ ,$$

while if the scalars are $\mathbb{C}$ then

$$\langle A\mathbf{a}, \mathbf{b} \rangle = \mathbf{b}^H A\mathbf{a} = (A^H \mathbf{b})^H \mathbf{a} = \langle \mathbf{a}, A^H \mathbf{b} \rangle \ .$$

So if $A = A^T$ or $A = A^H$ respectively then the relation

$$\langle A\mathbf{a}, \mathbf{b} \rangle = \langle \mathbf{a}, A\mathbf{b} \rangle \ .$$

holds—a "symmetric" relation, in which it doesn't matter on which side of the inner product one puts the matrix A.

(1) The notation $\langle \vec{a}, \vec{b} \rangle$ is preferable to $\vec{a} \bullet \vec{b}$ when one works with the inner product of functions, so we will adopt it immediately.

(2) Some people conjugate the first vector rather than the second in defining the complex inner product. Any such people in this course will have to make the obvious modifications for themselves. With the complex numbers as the scalar field, it really doesn't make that much difference. If one tries to extend this material to work with the (noncommutative) quaternions as the scalars, then one has to be more careful.

Recall the easy proof that eigenvectors belonging to distinct eigenvalues of a self-adjoint matrix must be orthogonal. First of all, the eigenvalues of a self-adjoint matrix must be real, because if $A\mathbf{x} = \lambda\mathbf{x}$ then

$$\begin{aligned} \lambda\langle \mathbf{x},\mathbf{x}\rangle &= \langle \lambda\mathbf{x},\mathbf{x}\rangle = \langle A\mathbf{x},\mathbf{x}\rangle = \langle \mathbf{x},A\mathbf{x}\rangle = \langle \mathbf{x},\lambda\mathbf{x}\rangle = \overline{\lambda}\langle \mathbf{x},\mathbf{x}\rangle \\ (\lambda-\overline{\lambda})\|\mathbf{x}\|^2 &= (\lambda-\overline{\lambda})\,\langle \mathbf{x},\mathbf{x}\rangle = 0\ . \end{aligned}$$

Thus if λ "really is an eigenvalue," with a nonzero $\mathbf{x}$, then also $\|\mathbf{x}\| > 0$ and one is forced to have $\lambda = \overline{\lambda}$, which is just a way of saying that λ is real.(3) Now with that detail out of the way, suppose A is self-adjoint (real or complex) and that $A\mathbf{x} = \lambda\mathbf{x}$ and $A\mathbf{y} = \mu\mathbf{y}$, with $\lambda \neq \mu$. Then essentially the same computation gives

$$\begin{aligned} \lambda\langle \mathbf{x},\mathbf{y}\rangle = \langle \lambda\mathbf{x},\mathbf{y}\rangle = \langle A\mathbf{x},\mathbf{y}\rangle = \langle \mathbf{x},A\mathbf{y}\rangle = \langle \mathbf{x},\mu\mathbf{y}\rangle = \overline{\mu}\langle \mathbf{x},\mathbf{y}\rangle = \mu\langle \mathbf{x},\mathbf{y}\rangle \\ (\lambda-\mu)\,\langle \mathbf{x},\mathbf{y}\rangle = 0\ . \end{aligned}$$

Under the assumption that $\lambda \neq \mu$ we are forced to have $\langle \mathbf{x},\mathbf{y}\rangle = 0$, that is, $\mathbf{x} \perp \mathbf{y}$.

In order to adapt that kind of reasoning to differential operators in the most useful way, we have to allow a slight generalization of the inner products that we all grew up with. Namely, if $\{m_1,\ldots,m_n\}$ is a sequence of positive "**weights**," then the **weighted inner product** that they define is the "dot product" given by

$$\langle \mathbf{a},\mathbf{b}\rangle_m = \sum_{j=1}^{n} m_j a_j b_j$$

or

$$\langle \mathbf{a},\mathbf{b}\rangle_m = \sum_{j=1}^{n} m_j a_j \overline{b}_j$$

for real and complex scalars respectively.(4) **Symmetry** and **self-adjointness** for matrices A **with respect to the** m**-weighted inner product** are then defined by requiring that the relation

$$\langle A\mathbf{a},\mathbf{b}\rangle_m = \langle \mathbf{a},A\mathbf{b}\rangle_m$$

hold for all vectors **a** and **b**. With a little effort, one can fight this notion of symmetry or self-adjointness back to a relation between A, A^T or A^H, and the diagonal matrix with the m_j's on the diagonal; it isn't worth the effort, however, because the defining relation does all the work. With $\|\mathbf{x}\|_m$ defined by $\|\mathbf{x}\|_m^2 = \langle \mathbf{x},\mathbf{x}\rangle_m$ one sees that "m-length" has all the properties we customarily associate with length, and—most important—our proof that symmetric and self-adjoint matrices have real eigenvalues and orthogonal eigenvectors goes through, provided only that we replace the usual Euclidean definition of perpendicularity by "$\mathbf{x} \perp_m \mathbf{y}$ if and only if $\langle \mathbf{x},\mathbf{y}\rangle_m = 0$."

2. Inner Products of Functions, and Self-Adjointness. I respectfully submit that it is not much of a step from the formula

$$\langle \mathbf{a},\mathbf{b}\rangle = \sum_{j=1}^{n} a_j\, b_j$$

to the definition

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}\,dx$$

(3) Note that even if we started with a real symmetric matrix and looked for complex eigenvectors, this argument would show that the complex eigenvalues—which always exist—would have to be real, and that therefore each would have some (nonzero) real eigenvectors belonging to it. The linear space of complex eigenvectors belonging to a given λ wili then be spanned by the real eigenvectors that it contains, as the reader may easily check; for the same statement with eigenfunctions, see Strauss, p. 117.

(4) Physics students who are used to the idea of diagonalizing two symmetric matrices simultaneously have probably seen these inner products before—they arise naturally when one looks at the kinetic energy of a finite system of masses.

that defines the **inner product of two functions** on an interval (which may be infinite at one or both ends). One thinks of the argument x as a "continuously varying index" and replaces "discrete summation" by "continuous summation," *i.e.*, integration. The "length" or **norm** of a function is then defined by

$$\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_a^b f(x)\overline{f(x)}\,dx} = \sqrt{\int_a^b |f(x)|^2\,dx}\ .$$

One ignores the complex-conjugation bar if one is dealing with real-valued functions. If $\|f\| = 0$ and f is continuous (or piecewise continuous, which is about the only case at which we shall look) then $f(x) = 0$ will hold everywhere on $[a, b]$ with possibly finitely many exceptions. The **Schwarz inequality**

$$|\langle f, g\rangle| \le \|f\|\,\|g\|$$

can be established by pretty much the same methods that work for vectors in $\mathbb{R}^n$ or $\mathbb{C}^n$. Two functions should be said to be **orthogonal** if $\langle f, g\rangle = 0$.

Much of Strauss's Ch. 5 is devoted to showing that the eigenfunctions of boundary problems for the second-derivative operator are orthogonal in the sense just defined. But as Strauss finally gets around to telling us in §**5.3**, these relations need not be derived from trigonometric identities: they follow from the fact that the second-derivative operator tries very hard to act on functions in the way that a self-adjoint matrix acts on vectors. Specifically: if f and g are two twice-continuously differentiable functions on the interval $[0, \ell]$, then[5]

$$\begin{aligned}
\langle -D^2 f, g\rangle &= -\int_0^\ell f''(x)g(x)\,dx = -\left\{ f'(x)g(x)\Big]_0^\ell - \int_0^\ell f'(x)g'(x)\,dx \right\}\\
&= f'(0)g(0) - f'(\ell)g(\ell) + \int_0^\ell f'(x)g'(x)\,dx \qquad (\$)\\
\langle -D^2 g, f\rangle &= g'(0)f(0) - g'(\ell)f(\ell) + \int_0^\ell f'(x)g'(x)\,dx \quad \text{and on subtraction}\\
\langle -D^2 f, g\rangle - \langle -D^2 g, f\rangle &= f'(0)g(0) - f'(\ell)g(\ell) - g'(0)f(0) + g'(\ell)f(\ell)\\
&= \begin{vmatrix} f(\ell) & g(\ell) \\ f'(\ell) & g'(\ell) \end{vmatrix} - \begin{vmatrix} f(0) & g(0) \\ f'(0) & g'(0) \end{vmatrix}\ .
\end{aligned}$$

If f and g satisfy end conditions that make those two determinants[6] zero (or even make their two values equal, so that cancellation occurs), then we shall have the relation

$$\langle -D^2 f, g\rangle = \langle -D^2 g, f\rangle = \langle f, (-D^2 g)\rangle \qquad (*)$$

which is **symmetry of the operator $-D^2$ with respect to the inner product of functions**. For example, if we suppose that both f and g satisfy the Dirichlet end conditions—so their values at $x = 0$ and $x = \ell$ are zero, or the Neumann end conditions—so the values of their derivatives at $x = 0$ and $x = \ell$ are zero—then the relation $(*)$ holds. The proofs that eigenvectors of a symmetric matrix are real and that eigenvectors belonging to different eigenvalues are orthogonal then go through in this context with virtually no change: for example, if $-D^2 f = \lambda f$ and $-D^2 g = \mu g$, then

$$\begin{aligned}
\lambda\langle f, g\rangle &= \langle -D^2 f, g\rangle = \langle -D^2 g, f\rangle = \langle \mu g, f\rangle = \mu\langle g, f\rangle\\
(\lambda - \mu)\langle f, g\rangle &= 0
\end{aligned}$$

[5] I have written out these relations without the complex-conjugation bar, because they are easier to follow that way. It is an easy exercise for the reader to go back and replace g by $\overline{g}$, etc., where appropriate—or the reader can simply see Strauss's discussion of the same matters on pp. 117–119.

[6] By the way, these determinants (or their transposes) are *Wronskians*; the reader has seen such objects before, in o. d. e. courses.

and again $\lambda \neq \mu$ forces $f \perp g$. Complex eigenvalues for $-D^2$ can also be ruled out in pretty much the same way we ruled them out for self-adjoint matrices: see Strauss, pp. 116–117, for the details (he gives essentially the same argument for orthogonality that we just gave, too, on pp. 116–117).

The relation labeled ($) above should not be viewed simply as an intermediate result. With $f = g$, it becomes

$$\langle -D^2 f, f\rangle = -\int_0^\ell f''(x)f(x)\,dx = f'(0)f(0) - f'(\ell)f(\ell) + \int_0^\ell f'(x)^2\,dx\ . \qquad (\$\$)$$

If f satisfies either the Dirichlet or the Neumann boundary conditions, then the two "integrated terms" are zero, and so if $f \not\equiv 0$ is an eigenfunction of $-D^2$ with eigenvalue λ, this relation becomes

$$\lambda\langle f, f\rangle = \langle \lambda f, f\rangle = \langle -D^2 f, f\rangle = -\int_0^\ell f''(x)f(x)\,dx = \int_0^\ell f'(x)^2\,dx \geq 0\ .$$

Thus one must have $\lambda \geq 0$, and $\lambda = 0$ is possible if and only if $f' \equiv 0$ is possible, *i.e.*, the constants are eigenfunctions—which means the boundary conditions were Neumann rather than Dirichlet. Without looking at specific solutions, we have ruled out the possiblity of negative eigenvalues, and forced positivity for the eigenvalues of the Dirichlet problem!

What about Robin conditions? These have the form $X'(0) - a_0X(0) = 0$ and $X'(\ell) + a_\ell X(\ell) = 0$. The first of these says that the vector $[X(0), X'(0)]$ is perpendicular (in $\mathbb{R}^2$) to the vector $[1, -a_0]$. If both $[f(0), f'(0)]$ and $[g(0), g'(0)]$ are perpendicular to $[1, a_0]$, then (we're in a two-dimensional space) they are proportional—the determinant $\begin{vmatrix} f(0) & g(0) \\ f'(0) & g'(0) \end{vmatrix} = 0$. The same argument is available at the endpoint $x = \ell$, making the determinant $\begin{vmatrix} f(\ell) & g(\ell) \\ f'(\ell) & g'(\ell) \end{vmatrix} = 0$. So the self-adjointness relation ($) holds for the Robin condition also, and eigenfunctions belonging to different eigenvalues are automatically orthogonal. How about nonnegativity of the eigenvalues? This question is trickier. Applying the Robin boundary conditions, we can substitute $a_0 f(0)$ for $f'(0)$ and $-a_\ell f(\ell)$ for $f(\ell)$ to make ($$) read[(7)]

$$\lambda\langle f, f\rangle = \langle -D^2 f, f\rangle = a_0 f(0)^2 + a_\ell f(\ell)^2 + \int_0^\ell f'(x)^2\,dx\ .$$

If the Robin conditions are radiative, so that $a_0 \geq 0$ and $a_\ell \geq 0$, then this relation again forces $\lambda \geq 0$, and in fact if $\lambda = 0$ then f must be constant and the constant must be zero unless both $a_0 = 0$ and $a_\ell = 0$, in which case we are simply back to Neumann boundary conditions. Thus all eigenvalues of $-D^2$ are strictly positive for radiative Robin boundary conditions at both ends.[(8)]

3. Weighted Inner Products of Functions, and Sturm-Liouville Differential Operators. It is not very hard to generalize the extremely useful orthogonality results of §2 above in the following way. Let $m(x) > 0$ and $p(x) > 0$ be functions on an interval $[a, b]$ (possibly infinite in one or both directions) that are strictly positive except perhaps at the endpoints; $p(x)$ should also be continuously differentiable. Let $q(x)$ be a continuous function on the interval. By analogy with the formula

$$\langle \mathbf{a}, \mathbf{b}\rangle = \sum_{j=1}^n m_j a_j b_j$$

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(7) Note that the r. h. s. of the formula is really an "energy" expression. Cf. Strauss's problem 11 (c) in section 4.3.

(8) It is possible to show, without explicit knowledge of the eigenfunctions, that there can only be finitely many nonpositive eigenvalues for Robin conditions in general. This statement is true generally for the regular Sturm-Liouville operators defined below. However, the only efficient proof of this fact that I know involves looking at the Green's function for the boundary-value problem, and we are not at that point yet!

for positive weights $\{m_j\}_{j=1}^n$, define the m-**weighted inner product**

$$\langle f, g\rangle_m = \int_a^b f(x)\overline{g(x)}\, m(x)\, dx$$

of the two functions f and g. This also defines an m-**weighted norm** by $\|f\|_m = \sqrt{\langle f, f\rangle_m}$, and it is clear that if $m(x) > 0$ throughout the interval and f is (say) piecewise continuous, then $\|f\|_m = 0$ implies that $f \equiv 0$ except perhaps at a discrete set of points. Rather than the operator $-D^2$, consider a general **Sturm-Liouville** (type) **differential operator** defined by

$$[Lf](x) = \frac{1}{m(x)}\left\{\frac{d}{dx}\left[-p(x)\,\frac{df}{dx}\right] + q(x)f(x)\right\}\ .$$

Then it is no harder to give the formal computations that demonstrated the relations $(*)$, etc., in this setting than in the setting of §2. If f and g are two twice-continuously differentiable functions on the interval $[a, b]$, then

$$\begin{aligned}
\langle Lf, g\rangle_m &= \int_a^b \frac{1}{m(x)}\left\{\frac{d}{dx}\left[-p(x)\,\frac{df}{dx}\right] + q(x)f(x)\right\} g(x)\, m(x)\, dx \\
&= -\int_a^b \frac{d}{dx}\left[p(x)\,\frac{df}{dx}\right] g(x)\, dx + \int_a^b q(x)f(x)g(x)\, dx \\
&= -\left\{ p(x)f'(x)g(x)\Big]_a^b - \int_a^b p(x)f'(x)g'(x)\, dx \right\} + \int_a^b q(x)f(x)g(x)\, dx \\
&= p(a)f'(a)g(a) - p(b)f'(b)g(b) \\
&\qquad + \int_a^b p(x)f'(x)g'(x)\, dx + \int_a^b q(x)f(x)g(x)\, dx. \text{ By symmetry,} \qquad (\$) \\
\langle Lg, f\rangle_m &= p(a)g'(a)f(a) - p(b)g'(b)f(b) + \int_a^b p(x)g'(x)f'(x)\, dx + \int_a^b q(x)g(x)f(x)\, dx \\
\langle Lf, g\rangle_m - \langle Lg, f\rangle_m &= p(a)f'(a)g(a) - p(b)f'(b)g(b) - p(a)g'(a)f(a) + p(b)g'(b)f(b) \\
&= p(b)\begin{vmatrix} f(b) & g(b) \\ f'(b) & g'(b) \end{vmatrix} - p(a)\begin{vmatrix} f(a) & g(a) \\ f'(a) & g'(a) \end{vmatrix}\ .
\end{aligned}$$

If the interval is a finite closed interval $[a, b]$ and the functions $m(x)$ and $p(x)$ are continuous and positive everywhere on the interval—so that $m(x) \geq \min\limits_{a\leq x\leq b} m(x) > 0$ and $p(x) \geq \min\limits_{a\leq x\leq b} p(x) > 0$—then the Sturm-Liouville operator is called **regular**, and the relation $\langle Lf, g\rangle_m = \langle f, Lg\rangle_m$ will hold if the two determinants are zero, or if $p(b)$-times the one at b has the same value as $p(a)$-times the one at a—the situation is no different from the one we encountered in §2 with the operator $-D^2$ (which is, of course, a regular S-L operator on a finite interval, with $p(x) \equiv m(x) \equiv 1$). If the operator is not regular (in which case it is called **singular**)—for example, if $m(x)$ or $p(x)$ is zero at an endpoint of $[a, b]$—then the question of "what is a boundary condition" becomes murkier. To give an important illustrative example: the (negative of the) Laplace operator in polar coördinates in 2 dimensions, for functions that depend on the polar radius r only, takes the form

$$-\nabla^2 f = -\frac{1}{r}\frac{d}{dr}\left[r\,\frac{df}{dr}\right]\ .$$

Here we would like to think of ∇^2 operating on functions $f(r)$ for $0 \leq r \leq R$, say. This fits into the S-L machine if we take $m(r) = r$ and $p(r) = r$, and the weighted inner product is natural because $\int_0^R f(r)g(r)\, r\, dr$ is, up to an inessential factor of 2π, the integral $\int_0^R \int_0^{2\pi} f(r)g(r)\, r\, d\theta\, dr = \iint_{\mathbb{D}_R} f(r)g(r)\, dA$ with respect to 2-dimensional area in the disc $\mathbb{D}_R$ of radius R. (From the standpoint of 2-dimensional space, then, we are dealing with an unweighted inner product of functions—but also with a p. d. e. boundary value problem rather than an o. d. e. boundary value problem.) However, the determinant condition at $r = 0$ becomes

$$p(0) \begin{vmatrix} f(0) & g(0) \\ f'(0) & g'(0) \end{vmatrix} = 0 \cdot \begin{vmatrix} f(0) & g(0) \\ f'(0) & g'(0) \end{vmatrix} = 0 \, ,$$

and there is no boundary condition on f and g at $r = 0$ except that they and their derivatives remain finite, so that one doesn't have to take limits as $r \to 0^+$ to find out what happens to this determinant term. (From the standpoint of seeking eigenfunctions of the Laplace operator on a disc in the plane, this finiteness requirement is certainly natural.) This **is** a boundary condition, but not like the ones we encountered with $-D^2$ on a finite interval of the real line. If (just to continue briefly with this example) we consider the eigenfunction equation

$$\lambda f = -\nabla^2 f = -\frac{1}{r}\frac{d}{dr}\left[r\,\frac{df}{dr}\right] \text{, or in a standard form}$$

$$r^2\,\frac{d^2 f}{dr^2} + r\,\frac{df}{dr} + \lambda r^2 f = 0 \, ,$$

it may not look quite as expected, because of the r^2 next to the λ. However, it works out quite nicely: the **Bessel function** $J_0(z)$ is **defined** to be the solution of the equation

$$z^2 u''(z) + z u'(z) + z^2 u(z) = 0$$

that has a power series expansion that converges for all z and that takes the value 1 at $z = 0$; these conditions determine $J_0(z)$ uniquely.[9] If we plug $J_0(\gamma r)$ into the l. h. s. of the eigenfunction equation above, we get

$$\begin{aligned} \gamma^2 r^2\, J_0''(\gamma r) + \gamma r\, J_0'(\gamma r) + \lambda r^2 J_0(\gamma r) &= \left[\gamma^2 r^2\, J_0''(\gamma r) + \gamma r\, J_0'(\gamma r) + \gamma^2 r^2 J_0(\gamma r)\right] - (\gamma^2 - \lambda) r^2 J_0(\gamma r) \\ &= (\lambda - \gamma^2) r^2 J_0(\gamma r) \, . \end{aligned}$$

Thus $J_0(\gamma r)$ will satisfy the eigenfunction equation for $\lambda = \gamma^2$; however, it will only satisfy the boundary conditions if γ is chosen correctly. For example, if we apply Dirichlet boundary conditions to the disc of radius $R = 1$, we would have to choose γ so that $J_0(\gamma) = 0$. Thus a list of eigenvalues and eigenfunctions for the Laplace operator on functions of radius only on the unit disc in $\mathbb{R}^2$, with the Dirichlet boundary condition $f(1) = 0$ (and the singular boundary condition that $f(r)$ and $f'(r)$ be finite at $r = 0$) is given by the functions $\{J_0(j_n r)\}_{n=1}^{\infty}$, where $\gamma_n = j_n$ is the n-th positive (real) zero of $J_0(z)$, counting up from $z = 0$. The corresponding eigenvalues of the negative of the Laplace operator are $\lambda_n = \gamma_n^2 = j_n^2$. It is not clear from this sketchy discussion—although it is true—that this is a complete list of the eigenfunctions of this singular Sturm-Liouville boundary value problem, or that every function of r satisfying suitable conditions can be expanded in a convergent series of these functions. However, we can prove in exactly the same way as on p. 4 above that all the eigenvalues of this Dirichlet problem are strictly positive, and that those of the corresponding Neumann problem (for which, it is easy to see, the γ_n's would be chosen as the zeros of $J_0'(\gamma)$, *i.e.*, as the consecutive minima and maxima of J_0) are nonnegative (but include $\lambda = 0$; the constant functions are eigenfunctions for the Neumann boundary conditions). If f satisfies either the Dirichlet or the Neumann boundary conditions, then the two "integrated terms" are zero—one because $p(r) = r$ is zero for $r = 0$, and the other from the boundary condition at $r = 1$—and so if $f \not\equiv 0$ is an eigenfunction of L with eigenvalue λ, one has the relation

$$\lambda \langle f, f \rangle = \langle \lambda f, f \rangle = \langle L f, f \rangle = \int_0^{\ell} f'(r)^2 \; r dr \geq 0 \, .$$

Thus one must have $\lambda \geq 0$, and $\lambda = 0$ is possible if and only if $f' \equiv 0$ is possible, *i.e.*, the constants are eigenfunctions—which means the boundary conditions were Neumann rather than Dirichlet. So again, without looking at specific solutions, we have ruled out the possiblity of negative eigenvalues, and forced positivity for the eigenvalues of the Dirichlet problem.

We shall be expanding on this sketch for the rest of the course.

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(9) See, for example, Michael D. Greenberg, *Advanced Engineering Mathematics*, 2nd ed., Prentice-Hall (1999), p. 230 ff. (and various other places) for some details. He also treats S-L problems in general, on p. 887 ff.