

TRANSPORT YET AGAIN

This is a cleaned-up version of the confusion I made on 9/6. My error—not mathematical, but certainly tactical—consisted in my failure to look at a general piece of the (liquid in the) pipe, corresponding to the interval $[x, x + \Delta x]$, looking as I did only at the interval $[0, \Delta x]$. While it's possible to patch up the derivation I did on the blackboard, it's better to start from scratch. We can also look at the “moving-frames” approach to the derivation, which is really the same but has a superior formulation: it turns out to be the coordinate method discussed on p. 7 of the text.

It should again be emphasized that there is nothing wrong with the derivation of the transport equation in Strauss's text. It's just that I think the “microscopic-in-space” tactic—the interval of integration $[x, x + \Delta x]$ will be made arbitrarily small in the treatment below—has advantages over the “macroscopic-in-space” approach taken by the textbook (his “ b ” can give you an arbitrarily large interval of integration in the first set-off equation on p. 11). The approaches shown here generalize to allow one to make some changes in the transport equation with ease and also to consider the case of diffusion-with-transport. My feeling is that it also gives one some intuition for the way the solutions of the (se various) equation(s) behave.

1. (a) Simple Transport à la pp. 10–11: The x -axis is laid out along the pipe carrying the substance. The water (or whatever) in the pipe is moving at velocity c along the pipe. For a given value of x , and at time $t = 0$, look at the amount of the substance in the interval $[x, x + \Delta x]$. After t units of time have elapsed, that substance will have been translated up the pipe by ct units of distance, but nothing else will have changed. So the integral

$$M = \int_{x+ct}^{x+\Delta x+ct} u(\tilde{x}, t) d\tilde{x} \quad (1.01)$$

(\tilde{x} is just a dummy variable of integration) is constant in time. Differentiating both sides of (1.01) with respect to t (using Leibniz's rule on the r. h. s.) gives

$$0 = \frac{dM}{dt} = -u(x + ct, t) \cdot c + u(x + \Delta x + ct, t) \cdot c + \int_{x+ct}^{x+\Delta x+ct} \frac{\partial u(\tilde{x}, t)}{\partial t} d\tilde{x} . \quad (1.02)$$

This equation holds for every x and every t , so in (1.02) we can now replace x by $x - ct$ wherever the former occurs. We see

$$0 = -cu(x, t) + cu(x + \Delta x, t) + \int_x^{x+\Delta x} \frac{\partial u(\tilde{x}, t)}{\partial t} d\tilde{x} . \quad (1.03)$$

Dividing both sides of (1.03) by Δx , we get

$$0 = c \cdot \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x} + \frac{1}{\Delta x} \int_x^{x+\Delta x} \frac{\partial u(\tilde{x}, t)}{\partial t} d\tilde{x} . \quad (1.04)$$

The choice of Δx was arbitrary, so we can take the limit as $\Delta x \rightarrow 0$ of both terms in (1.04). The limit of the first term is $c \cdot \frac{\partial u}{\partial x}$ by definition of $\frac{\partial}{\partial x}$ as the limit of difference quotients. The limit of the second term is the value of the integrand at $\tilde{x} = x$: you may either think of this as taking the limit of the average of the integrand over increasingly smaller intervals $[x, x + \Delta x]$, or think of it as the Fundamental Theorem of the Integral Calculus. In any event, we have in the limit

$$0 = c \cdot \frac{\partial u(x, t)}{\partial x} + \frac{\partial u(x, t)}{\partial t} , \quad \text{or} \\ \frac{\partial u(x, t)}{\partial t} + c \cdot \frac{\partial u(x, t)}{\partial x} = 0 \quad (1.05)$$

which is the transport equation in standard form.

1. (b) Simple Transport Combined with Other Stuff: Suppose that the substance being transported along the pipe was not stable. For two examples, suppose that it was a substance subject to radioactive

decay, or that it was a culture of bacteria subject to growth (of the kind that theoretical bacteria normally undergo). Then in the passage from (1.01) to (1.02) above we could not have said $\frac{dM}{dt} \equiv 0$; we would have had to apply the o. d. e. of growth or decay (respectively) and write

$$\frac{dM}{dt} = kM \quad (1.06)$$

where k is the rate of growth: $k > 0$ for bacteria, $k < 0$ for radioactive decay. But then we could simply go through the same computations that we did before, but now on both sides of the equal sign:

$$\begin{aligned} kM &= \frac{dM}{dt} = -u(x+ct, t) \cdot c + u(x+\Delta x+ct, t) \cdot c + \int_{x+ct}^{x+\Delta x+ct} \frac{\partial u(\tilde{x}, t)}{\partial t} d\tilde{x} \\ k \int_{x+ct}^{x+\Delta x+ct} u(\tilde{x}, t) d\tilde{x} &= -u(x+ct, t) \cdot c + u(x+\Delta x+ct, t) \cdot c + \int_{x+ct}^{x+\Delta x+ct} \frac{\partial u(\tilde{x}, t)}{\partial t} d\tilde{x} \\ k \int_x^{x+\Delta x} u(\tilde{x}, t) d\tilde{x} &= -u(x, t) \cdot c + u(x+\Delta x, t) \cdot c + \int_x^{x+\Delta x} \frac{\partial u(\tilde{x}, t)}{\partial t} d\tilde{x} \\ k \frac{1}{\Delta x} \int_x^{x+\Delta x} u(\tilde{x}, t) d\tilde{x} &= c \cdot \frac{u(x+\Delta x, t) - u(x, t)}{\Delta x} + \frac{1}{\Delta x} \int_x^{x+\Delta x} \frac{\partial u(\tilde{x}, t)}{\partial t} d\tilde{x} \\ ku(x, t) &= c \cdot \frac{\partial u(x, t)}{\partial x} + \frac{\partial u(x, t)}{\partial t} \quad \text{in the limit as } \Delta x \rightarrow 0; \text{ i.e.,} \\ \frac{\partial u(x, t)}{\partial t} + c \cdot \frac{\partial u(x, t)}{\partial x} - ku(x, t) &= 0 \quad \text{is the equation satisfied by the concentration function.} \end{aligned} \quad (1.07)$$

While this is a rather simple-minded model for the transport of (say) a radioactive solution, it can be used to gain intuition into the qualitative behavior of solutions of first-order linear p. d. e.'s. The case $c = 1$ and $k = -2$, for example, would lead to the p. d. e.

$$u_t + u_x + 2u = 0. \quad (1.08)$$

Now suppose that this equation had simply come to us *in abstracto*, with no reference to transport, radioactivity or anything else, but with the requirement that $u(x, 0) = f(x)$, a given function. The variables t and x enter symmetrically in this equation; however, if one wants to find a particular solution satisfying $u(x, 0) = f(x)$, then it makes sense to think of this as an **initial-value problem** with t as time and x as a space coordinate: even though the equation is abstract, we can use our quasi-physical intuition about transportation with decay. The function $f(x)$ then specifies an initial concentration. In class, we have looked abstractly at the “integrating-factor” approach to solving equations like this one: recall that multiplying the equation by e^{2t} turns it into

$$\frac{\partial [e^{2t}u]}{\partial t} + \frac{\partial [e^{2t}u]}{\partial x} = e^{2t} \left\{ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + 2u \right\} = 0 \quad (1.09)$$

so that if we set $v = e^{2t}u$ we can solve (1.09)—which will read

$$v_t + v_x = 0 \quad (1.10)$$

—by any method we know, then “solve for $u = e^{-2t}v$ ” to get the corresponding solution of (1.08), which is the equation in which we’re really interested. There is no mention of radioactive decay (and no “ u ” term) in (1.10), the “simple transport equation” which we understand fairly well either from the “geometric method” or from the “(change-of-)coordinate(s) method.” As we know, the solutions of (1.10) have the form $v(x, t) = g(x - t)$, where $g(\cdot)$ is an arbitrary (differentiable) function. The initial condition $u(x, 0) = f(x)$ turns into $v(x, 0) = e^{2 \cdot 0} \cdot u(x, 0) = f(x)$ and so we have

$$v(x, t) = f(x - t), \quad \text{thus} \quad u(x, t) = e^{-2t}f(x - t). \quad (1.11)$$

The form of the solution is plausible: the solution of the simple transport equation has simply been multiplied by a (purely-) time-dependent factor which is the one that goes with radioactive (exponential) decay at rate 2. The solution fits nicely with the interpretation of $f(x)$ as initial concentration. We see that if f is a bounded function (that is, there is some “large” number K for which $|f(x)| \leq K$ for all $x \in \mathbb{R}$) then $|u(x, t)| \leq Ke^{-2t}$ for all t , so that the solution is never larger than the largest of its values when $t = 0$, and tends to zero like e^{-2t} as $t \rightarrow +\infty$. This all makes sense from the standpoint of viewing the abstract equation as modeling transport with radioactive decay, so the simple quasi-physical model gives insight into the behavior of the solutions of the abstract equation. We shall see this sort of thing happen repeatedly (for the wave, diffusion = heat and Laplace equations) as the course progresses.

2. The Moving-Coördinate-Frame Approach: This is the “coördinate method” of p. 7 in disguise; it feels different only because the variable t is being interpreted as time. Let $u(x, t)$ give the density of the substance at x -coördinate x and time t . If the liquid is moving to the right in the pipe at constant velocity c and we install a new coördinate ξ whose origin moves to the right at the same velocity, then relative to the coördinates (ξ, t) the function u will be constant in time. Clearly (this is just the usual relation for translation of the origin of coördinates) $x = \xi + ct$. But relative to 2-dimensional (x, t) -space, what we are doing is installing new coördinates according to the linear transformation

$$\begin{aligned} \xi &= x - ct \\ \tau &= t \\ \text{Matrix} &= \begin{bmatrix} 1 & -c \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{2.01}$$

where to emphasize that there was a change of coördinates we introduced a new coördinate “ τ ”; it may look the same as t , but there is a real difference, as will be clear in what follows. Constancy in time says

$$\frac{\partial}{\partial \tau} u(\xi, \tau) = 0. \tag{2.02}$$

This is the equation satisfied by u in the (ξ, τ) coördinates, but we are now in the **inverse** of the situation of p. 7 of the text: we have the p. d. e. in the **new** coördinates, and we have to find out what it looks like back in the **old** coördinates (x, t) . So we need the inverse (matrix) of the linear transformation that was written out in (2.01):

$$\begin{aligned} x &= \xi + c\tau \\ t &= \tau \\ \text{Matrix} &= \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \end{aligned} \tag{2.03}$$

Recall (as we saw in class on 9/6, and as Strauss shows you rather un-explicitly in the first set-off equation following equation (6) on p. 29) that the differential operators transform by the **transpose** of the matrix (2.03) that transformed the coördinates:

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \frac{\partial}{\partial x} \\ \frac{\partial}{\partial \tau} &= c \cdot \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \\ \text{Matrix} &= \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \end{aligned} \tag{2.04}$$

(remember, we’re “going backward”: the p. d. e. was derived in the new coördinates, but we’re trying to see what it looks like in the old coördinates). In (x, t) -coördinates, the p. d. e. $u_\tau(\xi, \tau) = 0$ governing transport becomes the old familiar

$$0 = \frac{\partial}{\partial \tau} u(\xi, \tau) = c \cdot \frac{\partial u(x, t)}{\partial x} + \frac{\partial u(x, t)}{\partial t}; \tag{2.05}$$

anything that you can derive in this many ways must be right.

3. Diffusion with Transport: One can adapt the change-of-coördinates argument just given to the case of diffusion with transport (Problem §1.3 #5) to give a derivation of the corresponding p. d. e. that will perhaps be too glib for some people's taste.⁽¹⁾ The standard no-transport diffusion equation is

$$\frac{\partial u(x, t)}{\partial t} = k \frac{\partial^2 u(x, t)}{\partial x^2}. \quad (3.01)$$

If transport is occurring at velocity V (“ V ” is the “ c ” of §2 of these notes) and we replace the fixed (x, t) -coördinate frame by the moving (ξ, τ) -coördinate frame whose conversion formulæ are given in (2.01)–(2.04) above, then transport will be “invisible” because the origin of coördinates is moving, so that in (ξ, τ) -coördinates the diffusion equation remains

$$\frac{\partial u(\xi, \tau)}{\partial \tau} = k \frac{\partial^2 u(\xi, \tau)}{\partial \xi^2}. \quad (3.02)$$

We need a second differentiation in the space variable to convert this into (x, t) -coördinates, but that doesn't make much difference⁽²⁾ from the standpoint of the relations in (2.04): they simply extend to read

$$\begin{aligned} \frac{\partial}{\partial \xi} &= \frac{\partial}{\partial x} \\ \frac{\partial}{\partial \tau} &= V \cdot \frac{\partial}{\partial x} + \frac{\partial}{\partial t} \\ \frac{\partial^2}{\partial \xi^2} &= \frac{\partial^2}{\partial x^2} \\ \text{Matrix} &= \begin{bmatrix} 1 & 0 \\ V & 1 \end{bmatrix}. \end{aligned} \quad (3.03)$$

Thus back in (x, t) -coördinates the equation for diffusion with transport takes the form

$$\begin{aligned} \frac{\partial u(\xi, \tau)}{\partial \tau} &= k \frac{\partial^2 u(\xi, \tau)}{\partial \xi^2} \\ V \frac{\partial u(x, t)}{\partial x} + \frac{\partial u(x, t)}{\partial t} &= k \frac{\partial^2 u(x, t)}{\partial x^2} \\ \frac{\partial u(x, t)}{\partial t} &= k \frac{\partial^2 u(x, t)}{\partial x^2} - V \frac{\partial u(x, t)}{\partial x} \end{aligned}$$

and we're done. By the way, this approach also works to give a closed-form formula for the solution, once we have the “heat kernel” of Ch. 2. See Problem §2.4 #18.

(1) “Diffusion with transport” is the same as “heat conduction with convection.” See the problem referenced in the last paragraph above.

(2) But note that things would have been very much different for the second derivative in the “time” direction, if it had been required.