Problem set 3, Math 428, Fall 2007

From this problem set, hand in the problems The problems from the text and problems 3A, 3B, 3C concern degree sequences. Hand in 2.32, 2.34, 3B, and 3A on Thursday, September 27. Problems 3D,E,F are more challenging. Hand in one, of your choice, on October 4.

Problems from Chartrand and Zhang: 2.31, 2.32, 2.33, 2.34, 2.35, 2.36.

Additional problems:

3A. Let \((d_1, \ldots, d_n)\) be any sequence of non-negative integers whose sum is even. Show there is a multigraph \(G\) having \((d_1, \ldots, d_n)\) as a degree sequence. (Hint: use lots of loops.)

3B. The graph in Figure 1.10 on page 7 of the text has degree sequence \((5, 5, 5, 5, 4, 4, 2, 2, 2)\). In this graph, every vertex of degree 5 is connected by an edge to a vertex of degree 2. Use the method of the proof of Theorem 2.10, which involves switching edges, to construct a graph with the same degree sequence, but in which a vertex of degree 5 connects to vertices of degrees 5,5,4, and 4.

3C. The Erdős-Gallai criterions for graphicality that we will state in class says: The sequence \((d_1, \ldots, d_n)\), where \(d_1 \geq d_2 \geq \cdots \geq d_n\), is graphical if and only if \(\sum d_i\) is even and

\[
\sum_{k=1}^{n} d_i \leq k(k-1) + \sum_{k+1}^{n} \min\{k, d_i\} \quad \text{for each } 1 \leq k \leq n. \tag{1}
\]

a) Show if that if \((d_1, \ldots, d_n)\) satisfies condition (1), then \(d_i \leq n-1\) for all \(i\). (So condition (1) includes already this restriction on a graphical restriction.) Hint: apply (1) with \(k = 1\).

b) Does there exist a graph that has 13 vertices, 7 of which have degree 4 and three of which have degree 1, and has 31 edges? Use the Erdős-Gallai criterion along with the handshake lemma.

c) Prove the answer to part (b) directly, without using the Erdős-Gallai criterion.

3D. Show that if \((d_1, \ldots, d_n)\), with \(d_1 \geq d_2 \geq \cdots \geq d_n\) is the degree sequence of a graph, then the Erdős-Gallai condition—see (1) of problem 3C—must hold. (This proves the necessity part of the Erdős-Gallai theorem.)

3E. In problem 2.12, (see the solutions to Problem Set 2), we showed that the bound \(n-1\) was sharp by exhibiting a disconnected graph of order 5 for which \(\Delta(G) + \delta(G) = 3\). However, we claim that the following is true:

If \(G\) is connected graph of order \(n\) and if \(\Delta(G) + \delta(G) \geq n-2\), then \(\text{diam}(G) \leq 4\).

Prove this.

3F. In the answer to 2.10 c), (see the solutions to Problem Set 2), we showed that the bound in 2.10(b) is sharp because of the example of the empty graph on 3 vertices. However, we claim that:

If \(G\) is a graph of order strictly greater than 3 and if \(\text{deg}(u) + \text{deg}(v) \geq n-3\) for every pair of non-adjacent vertices, then \(G\) has at most two components.

Prove this.