2. Suppose $A$ and $S$ are non-empty sets (finite sets) of integers. Form a digraph $D$ in the following way: let $V(0) = A$ and draw a directed arc from $x$ to $y$ if $y \neq x$ and $y - x \in S$.

a) Draw $D$ if $A = \{0, 1, 2, 3, 4\}$ and $S = \{-2, 1, 2, 4\}$.

The directed edges are $01, 02, 04, 05, 12, 13, 21, 23, 24, 32, 34$.

Here is a picture; many different looking pictures are possible since the overall 'look' of the graph depends upon the position of the vertices.

b) Suppose $A$ and $S$ consist only of odd integers. What will the associated digraph look like?

Answer: The associated digraph will have no edges. If $x$ and $y$ are in $A$, both are odd, and so $y - x$ is even and does not belong to $S$.

c) What is the maximum size of $D$ if both $A$ and $S$ have 5 elements?

Recall that the size of a graph is the number of its edges. The maximum size in this case is 16. We will show this by first exhibiting sets $A$ and $S$ and an associated digraph of size 16. Then we will show no digraph associated to sets $A$ and $S$ of 5 elements each can have more than 16 edges.

(i) Let $A = \{0, 1, 2, 3, 4\}$. Let $S = \{-2, -1, 1, 2, 3\}$. Then one checks that the edges of $D$ are $01, 02, 04, 05, 12, 13, 21, 23, 24, 32, 34$; they are 16 in number.
(ii) We show that if \(|A| = |S| = 5\), \(D\) can have no more than 16 edges.

Let \(x_1, x_2, x_3, x_4, x_5\) be the 5 integers of \(A\) arranged so that \(x_1 < x_2 < x_3 < x_4 < x_5\). We will bound \(d_D(x_i)\) for each \(x_i\), considered as a vertex of \(D\).

- Observe first that

\[
x_1 - x_5 < x_1 - x_4 < x_1 - x_3 < x_1 - x_2 < x_2 - x_1 < x_3 - x_1 < x_4 - x_1 < x_5 - x_1,
\]

At most 5 of these 8 numbers can be in \(S\) and hence

\[d_D(x_i) \leq 5.\]

Similar arguments show \(d_D(x_5) \leq 5\).

- The following inequalities are true of \(x_2\):

\[
x_2 - x_5 < x_2 - x_4 < x_2 - x_3 < x_3 - x_2 < x_4 - x_2 < x_5 - x_2.
\]

Again, at most 5 of these can be in \(S\). If also \(x_1 - x_2 \in S\) and \(x_2 - x_1 \in S\), then \(d_D(x_2) \leq 7\).

Similarly \(d_D(x_4) \leq 7\).

- As \(x_3 - x_5 < x_3 - x_4 < x_4 - x_3 < x_5 - x_4\) and

\[
x_1 - x_3 < x_2 - x_3 < x_3 - x_2 < x_3 - x_1,
\]

it is possible for all eight of these numbers to be in \(S\), and so \(d_D(x_3) \leq 8\).

Using the handshake lemma,

\[
\text{size}(D) = \frac{1}{2} \sum_{i=1}^{5} d_D(x_i) \leq \frac{1}{2} (5 + 7 + 8 + 7 + 5) = 16.
\]
3 a) Let $S = \{1,2,3,4,5\}$. Let $G$ be the Petersen graph, that is, the graph whose vertices are the subsets of $S$ of size 2 and two vertices $\{1,2\}$ and $\{4,5\}$ are adjacent if and only if they are disjoint.

Write out the vertex set $V(G)$ and edge set $E(G)$ of $G$.

$V(G) = \{\{1,2\}, \{1,3\}, \{1,4\}, \{1,5\}, \{2,3\}, \{2,4\}, \{2,5\}, \{3,4\}, \{3,5\}, \{4,5\}\}$

$E(G) = \{\{1,2\}\{3,4\}, \{1,2\}\{3,5\}, \{1,2\}\{4,5\},\{1,3\}\{2,4\}, \{1,3\}\{2,5\}, \{1,3\}\{4,5\}, \{1,4\}\{2,3\}, \{1,4\}\{2,5\}, \{1,4\}\{3,5\}, \{1,5\}\{2,3\}, \{1,5\}\{2,4\}, \{1,5\}\{3,4\}, \{2,3\}\{4,5\}, \{2,4\}\{3,5\}, \{2,5\}\{3,4\}\}$

We know the list of $E(G)$ is complete because, since each vertex has degree 3, the handshake lemma implies $|E(G)| = \frac{1}{2} \sum_{v \in V(G)} d(v) = \frac{1}{2} (15 \cdot 3) = 15$ and our list has 15 different edges.

b) Represent the Petersen graph by a picture in the plane.

In a class of 15 students, each sends 3 Valentine cards to other students. How can this be modeled by a graph? Is it possible that each student receives cards from the same three students to whom he or she sent cards?

- This can be modeled by graphs in different ways; for example, if we are interested in every card sent, we can construct a digraph of 15 vertices, one for each student, and a directed edge from $x$ to $y$ if $x$ sends a card to $y$. 
However, if we are only interested in pairs of students who send each other cards and not in unrequited expressions of affection, we can use a graph whose vertices are again the students, but in which there is an undirected edge between $x$ and $y$ if and only if $x$ sends $y$ a card and $y$ sends $x$ a card. This latter graph is the appropriate one for answering the second question. If it is possible that each student receives cards from the three students to whom he or she sent cards, the degree of each vertex in the graph must be 3. But a graph of 15 vertices in which each vertex has degree 3 is not possible as the number of odd degree vertices in any graph is even.

5. In each case, give an example or explain why an example is not possible of order 7.

a) A graph with vertices of degrees 1, 1, 1, 2, 2, 3, 3.
   No such graph is possible because it would have to have an odd number of vertices of odd degree.

b) A graph with vertices of degrees 1, 2, 2, 2, 3, 3, 7. (Order 7 graph)
   No such graph with 7 vertices is possible because the maximum degree of a graph with 7 vertices is 6.

c) A graph whose vertices have degrees 1, 3, 3, 3, and has order 4
   Assume $G$ is a graph with 4 vertices $u_1, u_2, u_3, u_4$ and $d(u_1) = d(u_2) = d(u_3) = 3$. Then each of these vertices must be adjacent to $u_4$ and hence $d(u_4) = 3$ also. Hence no graph whose vertices have degrees 1, 3, 3, 3 is possible.

d) A graph with no odd vertices.
   Any cycle $C_n$ is an example. The degree of each vertex in a cycle is 2.
e) A graph (other than $K_4$ or the Petersen graph) in which each vertex has degree 3.
   Here is an example of order 6

   ![Graph Example]

f) A graph of order 5 such that no two adjacent vertices have the same degree.
   Here is an example:

   ![Degree Example]

  

  

  

g) A graph of order 5 or more such that any two non-adjacent vertices have different degrees:

   Here is an example; the integers next to the vertices are

   ![Example Graph]

   The vertex degrees.

   This example was chosen to give a connected graph with the required properties.

   The wise guy solution to this problem: $K_5$. There are no non-adjacent edges!

6. Give an example or show no example exists:

   a) A graph in which each vertex is adjacent to exactly two other vertices and each edge to exactly two other edges. Example:

   ![Example Graph]

   b) A graph of order 5 or more in which each vertex is incident to at least one edge but no two edges are adjacent. Example

   ![Example Graph]

   c) A graph of order 4 in which every two vertices are adjacent and every two edges are adjacent.

   This is not possible. A graph of order 4 in which every two vertices are adjacent must be $K_4$, but $K_4$ has non-adjacent edges.
7. Let \( S = \{s_1, \ldots, s_k\} \) and let \( E_1, \ldots, E_n \) be a collection of subsets of \( S \). We want to represent this structure by a graph. Here is one way. Let there be a vertex for each element of \( S \) and for each \( E_i \). \( V(G) = \{s_1, \ldots, s_k, E_1, \ldots, E_n\} \). Let \( s_j \in E_i \) be an edge if and only if \( s_j \in E_i \). Do not allow edges of the form \( s_j \in s_j \) or \( E_i \in E_i \).

Example \( S' = \{0, 1, 2, 3, 4\} \). \( E_1 = \{0, 1\} \), \( E_2 = \{0, 3, 4\} \), \( E_3 = \{1, 2, 4\} \).

\[
\begin{align*}
S' & \\
& \begin{array}{c}
0 & 1 & 2 & 3 & 4 \\
\end{array} \\
& \begin{array}{c}
E_1 & E_2 & E_3 \\
\end{array}
\]

8. Show that at least two vertices of a graph have the same degree. (Note: This is not true of multigraphs.)

Let \( G \) be a graph of order \( n \). The possible values of the degree of a vertex in \( G \) are the successive integers \( \{0, 1, 2, \ldots, n-1\} \). Since there are \( n \) integers in this set, a graph could only avoid having vertices of the same degree if there was a vertex of each of the degrees \( 0, 1, 2, \ldots, n-1 \). But this is not possible. The vertex of degree \( n-1 \) must have an edge to every one of the other \( n-1 \) vertices. But then no vertex could have degree 0. Thus there must be at least two vertices of the same degree.
7. Let \( S = \{ s_1, \ldots, s_K \} \). Let \( E_1, \ldots, E_n \) be a collection of subsets of \( S \). We want to represent this structure by a graph. Here is one way. Let \( G \) be the graph whose vertex set is

\[
V(G) = \{ s_{i_1}, \ldots, s_{i_k}, E_{i_1}, \ldots, E_{i_n} \} \quad \text{(a vertex for each element of } S \text{ and one for each set } E_i \}
\]

and for which the only edges are of the form \( s_{i_j}E_{i_k} \) where \( s_{i_j} \) is an element of \( E_{i_k} \). Thus the edges show which elements of \( S \) belong to which subsets.

Example

\( S = \{ 0, 1, 2, 3, 4 \} \quad E_1 = \{ 0, 1 \} \quad E_2 = \{ 0, 3, 4 \} \quad E_3 = \{ 1, 2, 4 \} \)

8. Show that at least two vertices in a graph have the same degree. (This is not true of multigraphs)

Consider a graph of order \( n \). Every vertex in a graph of order \( n \) has a degree less than or equal to \( n-1 \) and greater than or equal to 0. Since there are exactly \( n \) integers in the set \( \{0, 1, 2, \ldots, n-1\} \) in order for a graph not to have two vertices of equal degree