11.A. Prove: Let $A$ be a matrix of 0's and 1's. Then the maximum size of a set of entries 1 in this matrix, no two of which lie in the same row or column is equal to the minimum number of rows and columns that together contain all the 1's in $A$.

Define a bipartite graph $G$ with bipartition $X$ such that the vertices of $X$ are the rows of $A$, the vertices of $Y$ are the columns of $A$ and $ij$ is an edge if and only if the $(i, j)$ entry of $A$ is 1. Consider a matching $M$ in this graph. Each vertex of $X$ (each row of $A$) is incident to at most one edge of $M$, and similarly each vertex of $Y$ (each column of $A$) is incident to at most one edge of $M$. Therefore a matching represents a set of entries in the matrix of value 1, no two of which lie in the same row or column. Conversely, if we start with a set of entries of value 1 no two of which lie in the same row or column, the corresponding edges in the graph $G$ are a matching.

A vertex cover of $G$ is a subset of vertices such that every edge of $G$ is incident to at least one vertex of $G$. Since each edge represent an entry in $A$ of value 1, and the vertices represent rows and columns of $A$, a vertex cover yields a set of rows and columns which contain every entry entry of value 1.

The König-Egerváry theorem says that the size of a maximum matching in a bipartite graph equals the size of a minimum vertex cover. According to the interpretations of a matching and of a minimum vertex cover in $G$, this translates directly into the statement that the maximum size of a set of entries 1 in $A$, no two of which lie in the same row or column, equals the minimum number of rows and columns than contain all entries of value 1.

11.B. Suppose that $M_1$ and $M_2$ are two matchings in a bipartite graph with bipartition $(X, Y)$. Let $S$ be the $M_1$-saturated vertices of $X$ and let $T$ be the $M_2$-saturated vertices of $Y$. Show there is a matching $M$ that saturates $S$ and $T$ simultaneously.

The figure illustrates the situation; the $M_2$ matching is in bold edges, the $M_1$ in regular edges.
The solution to this problem will extract a matching \( M \) that saturates \( S \) and \( T \) simultaneously from the edges of \( M_1 \cup M_2 \).

It may be assumed that the only edges are those in \( M_1 \cup M_2 \). Observe then that, since \( M_1 \) and \( M_2 \) are matchings, at most two edges are incident to any vertex, and, if there are two edges, one is from \( M_1 \) and the other is from \( M_2 \). Thus, \textit{any} path in the graph of length two or more alternates between edges in \( M_1 \) and edges in \( M_2 \).

Start with some \( M \) matching in the graph. Suppose that there is a vertex \( x \) in \( S \) which is not saturated by \( M \). Consider the maximum length \( M \)-alternating path, \( P \), that starts at \( x \) and first traverses an edge of \( M_1 \). Since \( x \) is in \( S \) and is not saturated by \( M \), such \( M \)-alternating paths exist. By what we said above, this path alternates between edges in \( M_1 \) but not in \( M \) and edges in \( M_2 \) belonging to \( M \). There are three possible cases to consider. First the path \( P \) ends at a vertex in \( Y \). Then the path is \( M \)-augmenting, and by using this path to augment \( M \) one obtains a larger matching that saturates \( x \). Second, the path \( P \) ends in a vertex \( y \) of \( S \). But this cannot happen, because the edges along \( P \) alternate between those of \( M_1 \) and \( M_2 \) so \( P \) would enter \( y \) by an edge in \( M_2 \cap M \); but since \( y \) is in \( S \), there is also an edge of \( M_1 \) incident to \( x \) which could be used to extend \( P \), contradicting the maximality of \( P \). Finally, \( P \) could end in a vertex in \( X \) that does not belong to \( S \). In this case, modify \( M \) by removing the edges in \( P \) belonging to \( M \) and replacing them by the edges in \( P \) not belonging to \( M \). The new matching will saturate all the vertices of the path, including the initial vertex \( x \), except for the last vertex. But this last vertex does not belong to \( S \) and it is not necessary to saturate it.

In all cases then, we can build a new matching that saturates \( x \) and all vertices of \( S \cup T \) previously saturated by \( M \). By exactly the same reasoning, if \( y \) is not saturated by \( M \), we can build a new matching that saturates \( T \) and all vertices of \( S \cup T \) previously saturated by \( M \). Therefore the maximum subset of \( S \cup T \) that can be saturated by a matching drawn from \( M_1 \cup M_2 \) is \( S \cup T \) itself, and this is what we needed to prove.

\textbf{11C.} Prove Hall’s Theorem from the König-Egerváry theorem.

Hall’s theorem is: If \( G \) is a bipartite graph with bipartition \( (X,Y) \), then \( G \) contains a matching of \( X \) into \( Y \), that is a matching of cardinality \( |X| \) if and only if \( |N(S)| \geq |S| \) for every subset \( S \) of \( X \).

The necessity of the condition \( |N(S)| \geq |S| \) for every subset \( S \) of \( X \) is easy—see the first paragraph in the proof of Theorem 8.3 on page 186.
The more interesting direction is to assume $|N(S) \geq |S|$ for every subset $S$ and prove that $G$ contains a matching of $X$ into $Y$.

We shall do this by establishing that any vertex cover contains at least $|X|$ vertices. Since $X$ itself is a vertex cover, the minimum vertex cover size is $|X|$. Then the König-Egerváry theorem implies that $G$ contains a matching of size $|X|$ and any such matching certainly matches $|X|$ into $Y$, proving Hall's theorem.

Thus, let $U$ be any vertex cover. $G$ cannot contain any edges that join a vertex of $X - (U \cap X)$ to a vertex of $Y - (U \cap Y)$; such edges would not be covered by $U$ in contradiction to the assumption that $U$ is a vertex cover. Therefore $N(X - (U \cap X))$, the set of neighbors of $X - (U \cap X)$, must be contained in $U \cap Y$. By assumption, it follows that $|U \cap Y| \geq |X - (U \cap X)|$. Thus

$$|U| = |U \cap X| + |U \cap Y| \geq |U \cap X| + |X - (U \cap X)| = |X|.$$  

This completes the proof.

**11D.** Find a maximum matching in the following bipartite graph, starting from the matching shown. Prove your matching is maximum.

We show the matching here with labeled vertices

![Bipartite Graph](image)

First observe that $\{x_3, x_6, y_1, y_3, y_5, y_6, y_8\}$ is a vertex cover. This may be checked by examining each vertex that is not in this list and verifying that each of its edges terminates at the other end in a vertex in this list. This cover contains 7 vertices. Therefore if we obtain a matching using 7 edges, the König-Egerváry Theorem implies that the matching is maximum and that the vertex cover is minimum. Indeed there is a matching of size 7. To obtain it, start looking for $M$-augmenting path starting from the
unsaturated vertices of $X$, $X$ being the set of vertices on top. We find immediately that the edge $x_2y_1$ and the edge $x_6y_7$ are $M$-alternating paths of length one, that is, are edges that are not adjacent to any edges of the given matching. By adding these edges we obtain

$$
\begin{array}{cccccccc}
  x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\
  \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
  y_1 & y_2 & y_3 & y_4 & y_5 & y_6 & y_7 & y_8 \\
  \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ
\end{array}
$$

Although König-Egerváry tells us that this matching is maximum, we shall reprove this using the algorithm given in class notes for looking for $M$-augmenting paths or ruling them out.

Let $M$ denote the matching exhibited in the last graph. The first step of the algorithm is to label all $M$-unsaturated vertices of $X$ with a *. There is only one, namely $x_4$. The next step is to search $x_4$ and label with $x_4$ all vertices in $Y$ (the set of bottom vertices) reached by an edge not in $M$ from $x_4$. The vertices $y_5$ and $y_8$ receive this label. As they are both $M$-saturated, the algorithm continues and next searches $y_5$ and $y_6$ by following edges in $M$ to vertices in $X$. Thus $x_7$ receives label $y_5$ and $x_8$ receives label $y_8$.

As there are newly labeled vertices of $X$, they must next be searched. Taking $x_7$ first it is joined by edges not in $M$ to $y_3$ and $y_6$. These vertices have not yet been labeled so they receive label $x_7$. Then a search of $x_8$ leads $y_1$ to be labeled $x_8$. ($x_8$ is also joined to $y_6$ by an edge not in $M$ but $y_6$ was labeled in the previous step.) The result so far is shown below.
Now search \( y_1 \) and \( y_2 \), labeling \( x_1 \) by \( y_3 \) and \( x_2 \) by \( y_1 \). Next search the newly labeled vertices of \( X \). The edges not in \( M \) incident to \( x_1 \) connect it to already labeled vertices of \( Y \), and similarly for \( x_2 \). Therefore the procedure stops because no additional vertices are labeled. The final result is:

Examine now the final labeling. There is one \( M \)-unsaturated vertex in \( Y \), namely \( y_2 \), and it received no label. Therefore there are no \( M \)-augmenting paths and the matching \( M \) is maximum.

Notice that the labels can be traced back to determine \( M \)-alternating paths. We would probably only use this if we found that there is an \( M \)-augmenting path and we wanted to construct it so that we could augment
the matching. Although no $M$-augmenting path exists here, let us use the labels to find an $M$-alternating path joining $x_4$ to $x_2$, just to illustrate the procedure. So, since $x_2$ is labeled with $y_1$, trace back along the edge in $M$ to $y_1$, then from $y_1$ to the vertex with which it is labeled, namely $x_8$, and then to $y_8$, and finally to $x_4$. This gives the $M$-alternating path $x_2 y_1 x_8 y_8 x_4$. 