1.17  a) Prove that if $P$ and $Q$ are two longest paths in a connected graph, then $P$ and $Q$ have at least one vertex in common.

Proof by contradiction. Assume that $P$ and $Q$ are longest paths in a connected graph and that they do not share a common vertex. Then $P$ and $Q$ are the same length, call it $L$. Because the graph is connected there is a path $R$, one of whose end vertices is in $P$, such that the other end vertex is in $Q$ and such that all the other internal vertices of $R$ are in neither $P$ nor $Q$, as in the figure.

Let $P'$ and $P''$ be the subpaths of $P$ between the end vertices of $P$ and the vertex common to $P$ and $R$. Define $Q'$ and $Q''$ similarly. We can form 4 new paths: $P'RQ'$, $P'RQ''$, $P''RQ'$, and $P''RQ''$. At least one of $P'$ and $P''$ has length longer than $\frac{1}{2} (\text{Length } P) = \frac{1}{2} L$. At least one of $Q'$ and $Q''$ has length longer than $\frac{1}{2} L$ also. Since the length of $R$ is at least 1, one of these 4 new paths has length strictly greater than $L$. But this contradicts the assumption that $L$ is the longest path length. Hence it cannot be true that longest paths do not intersect.

b) Prove or disprove: If $P$ and $Q$ are two geodesics of length $k$ in a connected graph of diameter $k$, then $P$ and $Q$ have at least one vertex in common.

To disprove an assertion like this, it suffices to exhibit one
counter-example. Consider the graph \( G \). Its diameter is 2 and the paths \( u,w \) and \( y,z \) are geodesic but do not share a common vertex.

1.15 Draw all connected graphs of order 5 in which the distance between every two distinct vertices is odd.

There is only one such graph, \( K_5 \), the complete graph on 5 vertices, in which the distance between every pair of distinct vertices is 1.

Here is a sketch of a proof. (there are many ways to prove this). Let \( G \) be a graph of order 5 in which the distance between every two distinct vertices is odd.

(i) **Claim:** If \( v_1v_2v_3 \) is a path in \( G \), then \( v_1v_3 \) is an edge in \( G \). This must be true because if \( v_1v_3 \) were not an edge then it would have to be true that \( d(v_1,v_3)=2 \), which is not allowed.

(ii) **Claim:** If \( v_1v_2\ldots v_k \) is a path in \( G \) (\( k > 3 \)) then \( v_1v_k \) is an edge in \( G \). Prove this inductively starting with \( k = 3 \) as proved in (i).

We can use (ii) to finish the proof. Let \( u \) and \( v \) be any two distinct vertices in \( G \). Because \( G \) is connected there is a path from \( u \) to \( v \). By (ii) \( uv \) is an edge in \( G \). Hence there is an edge between every two vertices and so \( G = K_5 \).

**Question:** Did we use anything special about order 5 here? Or can we generalize this problem?
Let $P = u = v_0, v_1, \ldots, v_k = v$ be a $u$-$v$ geodesic in a graph $G$. Prove that $d(u, v_i) = i$ for each $i$, $1 \leq i \leq k$.

Recall that $d(u, v)$ is the length of the shortest path from $u$ to $v$, and that a geodesic is a shortest length path. Therefore $d(u, v_k) = k$ by definition.

It is worthwhile to solve this problem for $i < k$ by using a very important and basic fact about the distance function on vertices of a graph.

Claim (Triangle Inequality). If $u, v, w$ are vertices in a graph $G$ that is connected,

$$d(u, w) \leq d(u, v) + d(v, w).$$

Proof of claim. There is a path $P$ from $u$ to $v$ of length $d(u, v)$ and a path $Q$ from $v$ to $w$ of length $d(v, w)$. The walk obtained by first traversing $P$ and then $Q$ is a $(u, v)$ walk of length $d(u, v) + d(v, w)$. But we proved in class that a $(u, v)$ walk contains a $(u, v)$ path. Hence there is a $(u, v)$ path of length no more than $d(u, v) + d(v, w)$, which proves $d(u, w) \leq d(u, v) + d(v, w)$. 

Proof that $d(u, v_i) = i$ for $1 \leq i \leq k$.

Since $u = v_0, v_1, \ldots, v_i$ is a path of length $i$, $d(u, v_i) \leq i$.

Since $v_i, v_{i+1}, \ldots, v_k$ is a path of length $k - i$, $d(v_i, v_k) \leq k - i$.

By the triangle inequality $k = d(u, v_k) \leq d(u, v_i) + d(v_i, v_k)$.

Thus $d(u, v_i) = k - d(v_i, v_k) \geq k - (k - i) = i$. As we have shown $d(u, v_i) \leq i$ and $d(u, v_i) \geq i$, it follows that $d(u, v_i) = i$. 

1.20 a) Let \( u \) and \( v \) be distinct vertices in a connected graph.

What is the minimum size of a connected subgraph of
G containing \( u \) and \( v \)?

Answer: Any connected subgraph of \( G \) containing \( u \) and \( v \) must
contain a path from \( u \) to \( v \). So the minimum size of a connected
subgraph containing both \( u \) and \( v \) is at least \( d(u,v) \). On the
other hand, a geodesic (\( u,v \))-path is a connected subgroup
containing \( u \) and \( v \) and its size is \( d(u,v) \). Thus the minimum
size equals \( d(u,v) \).

1.22 Let \( G \) be a disconnected graph. Prove: if \( u \) and \( v \) are vertices
of \( \overline{G} \), then \( d_{\overline{G}}(u,v) = 1 \) or \( d_{\overline{G}}(u,v) = 2 \).

There are two cases: (i) \( u \) and \( v \) are in different connected components;
(ii) they are in the same connected component. In case (i), \( \overline{G} \) is not
in \( G \); hence \( uv \in \overline{G} \) and \( d_{\overline{G}}(u,v) = 1 \). In case (ii), either \( uv \) is
not in \( G \), in which case \( uv \in \overline{G} \) and \( d_{\overline{G}}(u,v) = 1 \), or \( uv \in G \).
If \( uv \in G \) let \( z \) be a vertex in a different connected component of \( G \); \( z \) exists
because \( G \) is disconnected by assumption. Then \( uz \) and \( vz \) are both
in \( \overline{G} \) and \( uzv \) is a \( (u,v) \)-path of length 2 in \( \overline{G} \), so \( d_{\overline{G}}(u,v) = 2 \).

1.23 Does there exist a connected graph \( G \) whose complement \( \overline{G} \) is
also connected and contains 4 distinct vertices \( u,v,x,y \) such that
\( d_G(u,v) = k = d_{\overline{G}}(x,y) \)?

\( a) \) Example with \( k = 1 \):

Example with \( k = 2 \):
b) Example with $k=3$. \[\begin{array}{c}
\begin{tikzpicture}
\node[shape=circle,draw=black] (x) at (0,0) {$x$};
\node[shape=circle,draw=black] (y) at (1,0) {$y$};
\draw (x) -- (y);
\end{tikzpicture}
\end{array}\] \[\begin{array}{c}
\begin{tikzpicture}
\node[shape=circle,draw=black] (x) at (0,0) {$x$};
\node[shape=circle,draw=black] (y) at (1,0) {$y$};
\draw (x) -- (y);
\end{tikzpicture}
\end{array}\]

There are no examples with $k > 3$. To see this it will suffice to study the subgraphs induced in $G$ by four distinct vertices $u, v, x, y$, and the corresponding subgraphs induced in $\overline{G}$.

Observe that if $uv \in E(G)$, $d_G(u,v) = 1$, and if $xy \notin E(G)$, $d_{\overline{G}}(x,y) = 1$, and, as we have treated already the case $k=1$, we need only consider induced subgraphs which include $xy$ as an edge, but not $uv$. We claim that in every case, either $d_G(u,v) \leq 3$ or $d_{\overline{G}}(x,y) \leq 3$. This implies $k = d_G(u,v) = d_{\overline{G}}(x,y)$ is not possible.

(i) If the size of the subgraph induced in $G$ by $\{u, v, x, y\}$ is one and $xy$ is an edge, the subgraph is \[\begin{array}{c}
\begin{tikzpicture}
\node[shape=circle,draw=black] (u) at (0,0) {$u$};
\node[shape=circle,draw=black] (v) at (1,0) {$v$};
\node[shape=circle,draw=black] (x) at (2,0) {$x$};
\node[shape=circle,draw=black] (y) at (3,0) {$y$};
\draw (u) -- (v);
\draw (x) -- (y);
\end{tikzpicture}
\end{array}\] Its complement is \[\begin{array}{c}
\begin{tikzpicture}
\node[shape=circle,draw=black] (u) at (0,0) {$u$};
\node[shape=circle,draw=black] (v) at (1,0) {$v$};
\node[shape=circle,draw=black] (x) at (2,0) {$x$};
\node[shape=circle,draw=black] (y) at (3,0) {$y$};
\draw (u) -- (x);
\draw (v) -- (y);
\end{tikzpicture}
\end{array}\] and hence $d_{\overline{G}}(x,y) = 2$.

(ii) If the subgraph includes $xy$ as an edge and has size 2 and if $uv$ is not an edge, it has either of the equivalent forms \[\begin{array}{c}
\begin{tikzpicture}
\node[shape=circle,draw=black] (u) at (0,0) {$u$};
\node[shape=circle,draw=black] (v) at (1,0) {$v$};
\node[shape=circle,draw=black] (x) at (2,0) {$x$};
\node[shape=circle,draw=black] (y) at (3,0) {$y$};
\draw (u) -- (v);
\draw (x) -- (y);
\end{tikzpicture}
\end{array}\] or \[\begin{array}{c}
\begin{tikzpicture}
\node[shape=circle,draw=black] (u) at (0,0) {$u$};
\node[shape=circle,draw=black] (v) at (1,0) {$v$};
\node[shape=circle,draw=black] (x) at (2,0) {$x$};
\node[shape=circle,draw=black] (y) at (3,0) {$y$};
\draw (u) -- (x);
\draw (v) -- (y);
\end{tikzpicture}
\end{array}\]. The complement of the first case is \[\begin{array}{c}
\begin{tikzpicture}
\node[shape=circle,draw=black] (u) at (0,0) {$u$};
\node[shape=circle,draw=black] (v) at (1,0) {$v$};
\node[shape=circle,draw=black] (x) at (2,0) {$x$};
\node[shape=circle,draw=black] (y) at (3,0) {$y$};
\draw (u) -- (x);
\draw (v) -- (y);
\end{tikzpicture}
\end{array}\] and again $d_{\overline{G}}(x,y) = 2$.

(iii) If the size is 3 and $xy$ is an edge but $uv$ is not, there are essentially 3 possibilities for which, respectively, $d_G(u,v) = 2$, $d_{\overline{G}}(x,y) = 2$, $d_G(u,v) \leq 3$.

(iv) If the size is 4 or 5 (and $xy$ is an edge but $uv$ is not), there must be edges adjacent to $u$ and to $v$ and then $d_G(u,v) \leq 3$. 
1.24. By applying the algorithm stated in class (see http://math.rutgers.edu/courses/428/428-f07/class-theorems.html) is bipartite and \( A = \{ q_1, r_1, u_1, y_1, w_1, t_1 \} \), \( B = \{ x_1, v_1, z_1, s_1 \} \) is a partition of the vertex set into independent sets.

Redrawing the graph

The graph is not bipartite because \( u_2 x_2 r_2 w_2 v_2 u_2 \) is an odd cycle.

1.27. The full solution is in the text.

1.28. a) Let \( R_n \) be the graph whose vertex set is the set of \( n \)-bit strings, where two vertices are adjacent if they differ in exactly two coordinates.

Picture of \( R_2 \):

\[
\begin{align*}
& (0,0) & & (0,1) \\
& (1,1) & & (1,0)
\end{align*}
\]

Picture of \( R_3 \):

\[
\begin{align*}
& (0,0,0) & & (1,1,0) & & (0,0,1) & & (1,0,0) \\
& (0,0,1) & & (1,1,1) & & (0,1,0) & & (1,0,1) \\
& (0,1,1) & & (1,0,1) & & (0,1,1) & & (1,0,1)
\end{align*}
\]
b) Let $S_n$ be the graph whose vertices are $n$ bit strings, two vertices being adjacent if they differ in exactly 3 coordinates. $S_2$ does not make sense, so we draw $S_3$.

```
(0,0,0) -- (0,1,0) -- (1,0,0) -- (1,1,0) ----
|                  |                  |                  |
(1,1,1) -- (1,0,1) -- (0,1,1) -- (0,0,1) ----
```

2.6 Suppose $G$ is a graph of order $3n$ with $n$ vertices of each of the degrees $n-1, n, n+1$. Prove $n$ is even.

By Theorem 2.4, $n(n-1) + n(n-1) + n(n+1) = 3n^2$ is even. This can only be true if $n$ itself is even.

2.8 Let $G$ be a graph of order $n$. If $\deg u + \deg v + \deg w \geq n-1$ for every three pairwise non-adjacent vertices $u, v,$ and $w$, must $G$ be connected?

No: **Counterexample A**

\[ G: \begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array} \]

In this example there is no triple of pairwise non-adjacent vertices so the condition $\deg u + \deg v + \deg w \geq n-1$ for every three pairwise non-adjacent vertices is vacuously true.

**Counterexample B**

\[ G: \begin{array}{ccc}
\circ & \circ & \circ \\
\circ & \circ & \circ \\
\end{array} \]

Here $G$ has order 5, $u, v, w$ are pairwise non-adjacent, $\deg u + \deg w + \deg v = 4 = 5 - 1$, but $G$ is not connected.
2.10 a) Show there exists a connected graph of order n such that \( \deg u + \deg v \geq n-2 \) for every two non-adjacent vertices \( u \) and \( v \) and for which \( \deg x + \deg y = n-2 \) for some pair of non-adjacent vertices.

For the condition to be non-vacuous, assume \( n \geq 3 \). There are many different ways to construct examples. Here is one. Let \( G \) be the graph formed by adding a vertex \( x \) and an edge from \( x \) to \( y \) of the graph \( K_{n-1} - yw \), where \( yw \) is an edge in \( K_{n-1} \).

Then \( \deg x = 1 \) and \( \deg y = n-3 \)
while \( \deg u = n-2 \) for all other \( u \)
\( x \) and \( y \) are not adjacent and
\( \deg x + \deg y = n-2 \). Clearly \( \deg u + \deg v \geq n-2 \) for all pairs of vertices in this graph, adjacent or not.

b) Prove that if \( \deg u + \deg v \geq n-2 \) for every pair of non-adjacent vertices, then \( G \) has at most two components. \([G \text{ is of order } n] \)

Assume that \( G \) has at least two components and that \( u \) and \( v \) are in different components. Then \( u \) and \( v \) are not adjacent and so \( \deg u + \deg v \geq n-2 \). Because \( u \) and \( v \) are in different components, they do not share any common neighbors (two vertices are neighbors if there is an edge between them). The number of neighbors of \( u \) is \( \deg u \), of \( v \) is \( \deg v \) and since there are precisely \( n-2 \) other vertices in \( G \) other than \( u \) and \( v \) and since \( \deg u + \deg v \geq n-2 \), it must in fact be true that \( \deg u + \deg v = n-2 \) and each vertex in \( G \) is either a neighbor of \( u \) -- in which case it is in the same component as \( u \), or a neighbor of \( v \) -- in which
case it is the same component as \( v \). Thus \( G \) has at most 2 components.

c) Is the bound in b) sharp? To say the bound is sharp is to say there exists a graph of some order \( n \) with 3 components such that \( \deg u + \deg v \geq n - 3 \) for all non-adjacent vertices \( u \) and \( v \). Indeed there is such an \( n \) and such a graph: for \( n = 3 \), the empty graph \( \boxed{\cdot \cdot \cdot} \) on 3 vertices satisfies \( \deg u + \deg v \geq 0 = n - 3 \) for all \( u, v \). However, this is the only example; if \( n > 3 \) and \( \deg u + \deg v \geq n - 3 \) for all non-adjacent \( u, v \), then \( G \) has at most two components. We leave the proof as an exercise.

2.12 Prove: If \( G \) is a graph of order \( n \) and \( \Delta(G) + \delta(G) \geq n - 1 \), then \( G \) is connected and \( \text{diam}(G) \leq 4 \). Show that the bound \( n - 1 \) is not sharp.

- The graph \( \boxed{\cdot \cdot \cdot} \) shows the bound is sharp; it is connected and \( \Delta(G) + \delta(G) = 2 + 1 = 5 - 2 \) (here \( n = 5 \)).
- Let \( u \) be a vertex of maximum degree; \( \deg u = \Delta(G) \).

The proof will follow easily from the claim. For every vertex \( x \) in \( G \), \( x \) is connected to \( u \) and \( d(x, u) \leq 2 \). To see how the proof follows observe first that if \( x \) and \( y \) are any two vertices, \( x \) is connected to \( y \) because \( x \) is connected to \( u \) and \( u \) is connected to \( y \).

Also, using the triangle inequality discussed in the solution to problem 1.16, \( d(x, y) \leq d(x, u) + d(u, y) = 2 + 2 = 4 \). Hence \( \text{diam}(G) \leq 4 \).

To prove the claim, let \( x \) be any vertex of \( G \) other than \( u \). Then \( \deg x + \deg u \geq \Delta(G) + \delta(G) \geq n - 1 \) since \( \deg x \geq \delta(G) = \text{minimum degree} \) and \( \deg u = \Delta(G) \). But then, using the proof of Theorem 2.4, either \( xu \) is an edge or \( x \) and \( u \) are both adjacent to some other vertex \( w \). In the latter case \( xuw \) is an \( (x, u) \) path of length 2 so \( x \) and \( u \) are connected and \( d(x, u) \leq 2 \).