2.19 a) Construct an \( r \)-regular graph of order 6 for all possible \( r \).

By Theorem 2.6, since 6 is even, there are \( r \)-regular graphs of order 6 for all orders \( r \) up to the maximum \( r = 5 \). A list of examples follows.

- \( r = 0 \): \( K_6 \), the empty graph on 6 vertices
- \( r = 1 \): \( 3K_2 \), or, in a picture, 
- \( r = 2 \): the Hanay graph \( H_{2,6} \), which is the same as \( C_6 \)
- \( r = 3 \): the Hanay graph \( H_{3,6} \)
- \( r = 4 \): the Hanay graph \( H_{4,6} \)
- \( r = 5 \): \( K_6 \), the complete graph on 6 vertices

b) Repeat a) for order 7. In this case, \( r \) must be even.

- \( r = 0 \): \( K_7 \), the empty graph on 7 vertices is the only example
- \( r = 2 \): \( H_{2,7} = C_7 \), the cycle with 7 vertices
- \( r = 4 \): \( H_{4,7} \)
- \( r = 6 \): \( K_7 \).

2.22 a) Construct a 3-regular graph containing \( G \) by the method of Theorem 2.7.

Step 1 Draw 2 copies of \( G \) and connect corresponding vertices whose degrees in \( G \) are less than 3.
Step 2. Because 2 vertices of degree 2 remain, repeat the procedure.

b) Construct a 3-regular graph $H$ of minimal order that contains $G$ as an induced subgraph.

Suppose $H$ is a 3-regular graph that contains $G$ as an induced subgraph. $G$ has a vertex of degree 1. This vertex must have 2 neighbors in $H$ which are not vertices of $G$, and thus order $(H) = \text{order}(G) + 2 = 7$. However, a 3-regular graph must be of even order and so order $(H) \geq 8$, at least. In fact, there exist graphs $H$ of order 8 that include $G$ as an induced subgraph. Here is an example.

2.24 What is the minimum order of a 3-regular graph $H$ containing $G$ as an induced subgraph, the graph $G$:

This problem is similar to 2.22. Again because $G$ is order 5 and contains vertices of degree 1, any 3-regular graph containing $G$ as an induced subgraph must have at least 8 vertices. To show that 8 is indeed the minimum order, we must exhibit an $H$ of order 8 containing $G$ as an induced subgraph. Here it is.
4A. List all graphs of size 5 or less of order 5, up to isomorphism.

The main problem in doing this exercise is to have an organized procedure that ensures we list all possible graphs. Here we proceed in order of increasing size and build the graphs of size $k+1$ from those of size $k$ as follows. Take a graph of size $k$ and add an edge; if the new graph is not isomorphic to a graph already in our list, add it to the list. Continue in this manner adding new untried edges until all possibilities on the size $k$ graphs are exhausted. It is not actually necessary to carry out this procedure exhaustively, especially for graphs of size 0, 1, 2, 3. And for sizes 4 and 5 many possibilities can be rejected quickly without drawing.

In this list we also write down the degree sequence of each graph.

<table>
<thead>
<tr>
<th>SIZE 0</th>
<th>SIZE 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_0$:</td>
<td>$G_1$:</td>
</tr>
<tr>
<td>(0,0,0,0,0)</td>
<td>(1,1,0,0,0)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SIZE 2</th>
<th>SIZE 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_3$: (2,1,1,0,0)</td>
<td>$G_4$: (1,1,1,1,0)</td>
</tr>
<tr>
<td>$G_5$: (2,2,2,0,0)</td>
<td>$G_6$: (2,2,1,1,0)</td>
</tr>
<tr>
<td>$G_7$: (2,1,1,0,0)</td>
<td>$G_8$: (3,1,1,1,0)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SIZE 4</th>
<th>SIZE 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Adding an edge in different ways to $G_5$ yields</td>
<td>Adding different edges to $G_6$ yields</td>
</tr>
<tr>
<td>$G_9$: (3,2,2,1,0)</td>
<td>$G_{10}$: (2,2,2,1,1)</td>
</tr>
<tr>
<td>$G_{11}$: (2,2,2,2,0)</td>
<td>$G_{12}$: (2,2,2,1,1)</td>
</tr>
<tr>
<td>$G_{13}$: (3,2,2,1,1)</td>
<td></td>
</tr>
</tbody>
</table>

...
SIZE 4 Continued. Adding an edge to G₈ produces just one new graph.

\[ G_{14} \]
\[ (4,1,1,1,1) \]

SIZE 5: The first series is produced by adding edges to G₉:

\[ G_{15} \]
\[ (3,2,2,2,1) \]

\[ G_{16} \]
\[ (4,2,2,1,1) \]

\[ G_{17} \]
\[ (3,3,2,2,0) \]

\[ G_{18} \]
\[ (3,3,2,1,1) \]

\[ G_{19} \]
\[ (3,3,2,2,0) \]

Notes: The 6th possible graph one gets from G₉ is \[ \boxed{G_{10}} \] but this is isomorphic to G₁₈. Of the graphs G₁₅ - G₁₉ only G₁₇ and G₁₉ have the same degree sequence and G₁₇ \( \neq \) G₁₉ because the degree 3 vertices are adjacent in G₁₇ but not in G₁₉. Thus G₁₅ - G₁₉ are truly mutually non-isomorphic.

Any edge added to G₁₀ yields a graph isomorphic to G₁₅.

Adding an edge to G₁₁ yields either a graph isomorphic to G₁₉ or the new graph \[ G_{20} \]
\[ (3,2,2,2,1) \]
(This is not isomorphic to G₁₅ because it contains no triangle.)

From G₁₂ one obtains the cycle \[ G_{21} \]
\[ (2,2,2,2,2) \]

No new graphs are obtained by adding an edge anywhere to G₁₃ or G₁₄.
The last part of 4A asks for any self-complementary graphs necessarily of size 5, in the list above.

Recall that if $G$ is self-complementary and $(d_1, d_2, d_3, d_4, d_5)$ is its degree sequence, $(4-d_1, 4-d_2, 4-d_3, 4-d_4, 4-d_5)$ must, after proper rearrangement be the same as $(d_1, d_2, d_3, d_4, d_5)$ since $(4-d_1, \ldots, 4-d_5)$ is the degree sequence of $\overline{G}$.

Only the degree sequences of $G_{15}, G_{18}, \ldots, G_{26}$ and $G_{21}$ satisfy this condition. The complement of $G_{15}$ is isomorphic to $G_{26}$, so $G_{15}$ and $G_{26}$ are not self-complementary. $G_{18}$ and $G_{21}$ must be self-complementary if our list is correct because they are the only graphs in the list with their degree sequences. To double check we draw $\overline{G}_{18}$ and $\overline{G}_{21}$ and exhibit the isomorphism.

\[
\begin{align*}
\text{Isomorphism } & \phi(a) = a' \\
\phi(b) &= b' \\
\text{etc.}
\end{align*}
\]
3.2. Give an example of 3 graphs of the same order, same size and same degree sequence such that no two are isomorphic.

There are many possibilities. Here is one:

\[ G_0 = 3C_3 \]
\[ G_1 = C_3 \cup C_6 \]
\[ G_2 = C_4 \cup C_5 \]

The degree sequence of each of these graphs on 9 vertices is \( (2, 2, 2, 2, 2, 2, 2, 2, 2) \).

Here is an example with graphs of order 6:

\[ H_0 : \]
\[ \begin{array}{cccccc}
1 & 2 & 3 & 2 & 1 \\
3 & 2 & 2 & 1 & 0
\end{array} \]

\[ H_1 : \]
\[ \begin{array}{cccccc}
1 & 2 & 3 & 2 & 1 \\
0 & 0 & 0 & 0 & 0
\end{array} \]

\[ H_2 : \]
\[ \begin{array}{cccccc}
2 & 3 & 1 & 1 & 0 \\
2 & 2 & 2 & 2 & 0
\end{array} \]

There are no examples of order 5, as one can see from studying the list in problem 4A.

3.4. by the isomorphism \( \phi(a) = a' \)
\( \phi(b) = b', \) etc.

The isomorphism again is \( \phi(a) = a', \phi(b) = b', \) etc.
3.8 $G_1$, $G_2$, $G_3$

$G_1$ and $G_3$ are not isomorphic because $G_1$ contains $C_4$ but $G_3$ does not.

$G_1$ and $G_2$ are isomorphic -- labels giving the isomorphism are below.

Because $G_1 \cong G_2$, $G_2$ cannot be isomorphic to $G_3$.

3.16 The complement of a graph of order 9 with degree sequence $(6, 6, 6, 6, 6, 6, 6, 6, 6)$ has degree sequence $(2, 2, 2, 2, 2, 2, 2, 2, 2)$. Theorem 3.1 says that two graphs are isomorphic if and only if their complements are isomorphic. Therefore the number of non-isomorphic 6-regular graphs equals the number of non-isomorphic 2-regular graphs and we know from the text -- see page 65, Figure 3.16 -- that there are 4 such graphs.

Remark: How do we know, rigorously, that Figure 3.16 is a complete listing, up to isomorphism, of 2-regular graphs of order 9? Here is a theorem that you should prove as an exercise -- an induction argument works and it helps to use one of the class theorems.

**Theorem**: Every 2-regular graph is a disjoint union of cycles.