Problem Solutions, 9A-9D  Math 428

9A. Claim: For the graph $G$,

$$\chi(G) = \lambda(G) = 3$$

Proof. By Whitney's theorem

$$\chi(G) \leq \lambda(G) \leq \delta(G) = 3.$$ 

Therefore to finish the proof it is only necessary to show $\chi(G) = 3$. Since adding edges to a graph can not decrease its connectivity, it will suffice to show $\chi(G') = 3$ for the subgraph $G'$ drawn below.

First observe that there are 3 internally disjoint paths between any two distinct vertices of the subgraph $G''$ of $G'$ induced by vertices $a, b, c, d, e, f, g$:

This is not hard to see. Consider any two vertices on the outer cycle $bcdefgb$ of $G''$. The cycle yields two internally disjoint paths between two vertices on the outer cycle, and there is a third path that use only edges incident to vertex $a$. There are also 3 internally disjoint paths from $a$ to any other vertex. By symmetry it suffices to check this for paths from $a$ to $c$ and from $a$ to $d$ and this is easily done by inspection. Menger's theorem implies $G''$ is 3-connected.

Now $G'$ is obtained by adjoining vertex $h$ with edges to three vertices of $G''$. Since $G''$ is 3-connected, Corollary 5.18 implies
$G'$ is 3-connected. Thus $G$ is 3-connected and since we know $\chi(G) \leq 3$, it follows $\chi(G) = 3$.

9B. The sequences $e_1e_2e_3$, $f_1f_2f_3f_4f_5$, and $g_1g_2g_3g_4$ define 3 edge-disjoint $(x,y)$-paths. At the same time, $\{h_1, h_2, h_3\}$ is an edge cut separating $x$ and $y$. By Menger’s theorem, the maximum number of edge-disjoint $(x,y)$-paths is 3.

9C. (i) $K_2$ is the only non-separable graph of order 2.

(ii) $K_3$ is the only non-separable graph of order 3.

(iii) The non-separable graphs of order 4 are $C_4$ and what is obtained from $C_4$ by adding edges: $K_4$.

(iv) There are 40 non-separable graphs of order 5. In the first series, we start with $C_5$ and add edges in all possible ways until we arrive at $K_5$:

The others are:

b) Consider a block decomposition of a graph. Imagine building the graph by adding blocks one at a time, always connecting the new block to a connected subgraph of the previously inserted blocks. Except for the first block, each addition adds $n_i-1$ vertices where $n_i$ is the number of vertices in block $i$. Using this principle,
one cannot build a graph of order 7 with 3 blocks of order 2 or odd orders 3, 2, and 2. The possible combinations for a connected graph are:

- 3 blocks of order 3, which may be arranged in 2 ways: linearly as \( \circ \circ \circ \), or as a "flower" \( \circ \circ \circ \).
- a block of order 4, of order 3 and of order 2, which have 3 possible linear arrangements \( \begin{array}{c} 4 \circ 3 \circ 2 \\ 3 \circ 2 \circ 4 \end{array} \) and a flower arrangement.
- a block of order 5 and 2 blocks of order 2, which have 2 possible linear arrangements \( \begin{array}{c} 5 \circ 2 \circ 2 \\ 2 \circ 3 \circ 5 \end{array} \) and a flower arrangement.

If the graph is not connected, we can admit the combinations

\[
\begin{array}{c}
\circ \circ \circ \circ \circ \\
3 \circ 3 \circ 2 \\
\circ \circ \circ \circ \circ \\
3 \circ 3 \circ 2 \\
\circ \circ \circ \circ \circ \\
2 \circ 2 \circ 3
\end{array}
\]

(c) How many graphs are there, up to isomorphism, of order 7, size 7, and having 3 blocks?

Connected case:

If a graph is connected and has order 7 and size 7 it can admit at most one cycle and must have exactly one cycle. Since each non-separable graph of orders 3, 4 and 5 contains a cycle, a connected example can have at most one block of order 3 or more. Among the possibilities listed above, this leaves only the combination of a block of order 5 with 2 blocks of order 2. Moreover, inspection of the non-separable blocks of order 5 shows that \( C_5 \) is the only one with just one cycle. Thus the block of order 5 must be \( C_5 \).
We have the following possibilities up to isomorphism:

![Graphs](image)

The non-connected possibilities are

![Graphs](image)

Note: This problem would have been better written with a restriction to connected graphs, but I failed to make that distinction in the problem statement and so I included here the case of disconnected graphs. Note that by definition trivial graphs with only one vertex are not blocks.

9.4. What are $K(K_{4,8})$ and $\lambda(K_{4,8})$?

Let each vertex of $K_{4,8}$ have either degree 8, if it belongs to the partite set of 4 vertices, or degree 4, if it belongs to the partite set of 8 vertices. Hence $\delta(K_{4,8}) = 4$ and $K(K_{4,8}) \leq \lambda(K_{4,8}) \leq 7$ by Whitney's theorem.
We claim that in fact $\chi(K_{4,8}) = 4$. It follows that $\Delta(K_{4,8}) = 4$ as well.

To show $\chi(K_{4,8}) = 4$, it is enough to show that $K_{4,8} - S$ is connected whenever $S$ is a set of vertices of cardinality 3 or less. Let $X$ and $Y$ denote the vertex sets of the bipartition of $K_{4,8}$, with $|X| = 4$ and $|Y| = 8$.

Since $1S < |X| < |Y|$, $X - (X \cap S)$ and $Y - (Y \cap S)$ are both non-empty. But $K_{4,8} - S$ is the complete bipartite graph on the bipartition $X - (X \cap S)$, $Y - (Y \cap S)$ because the removal of the vertices in $S$ does not remove edges connecting vertices in $X - (X \cap S)$ to those in $Y - (Y \cap S)$. As any complete bipartite graph is connected, $K_{4,8} - S$ is connected.