Math 436 homework solutions (chapter 7)

4 To solve $x+y=10$, $x/y+y/x=13/6$ $(0 \le x, y \le 10)$.
Substitute $y=10-x$ to get $x/(10-x)+(10-x)/x=13/6$; clearing denominators yields $0=x^2-10x+24=(x-4)(x-6)$ so the solution is $x=4, y=6$.

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$$
\begin{array}{rl}
 & 10x^3 + \quad x^2 + \ 4x + \ 10 + \quad 0/x + \quad 8/x^2 + 2/x^3 \\
\hline
2x^3+5x+5+\frac{10}{x} & |20x^6+2x^5+58x^4+75x^3+125x^2+96x+94+\frac{140}{x}+\frac{50}{x^2}+\frac{90}{x^3}+\frac{20}{x^4} \\
 & \underline{20x^6+0x^5+50x^4+50x^3+100x^2} \\
 & \qquad 2x^5+8x^4+25x^3+25x^2+96x+94+\frac{140}{x}+\cdots+\frac{20}{x^4} \\
 & \qquad \underline{-2x^5+0x^4+5x^3+5x^2+10x} \\
 & \qquad\qquad 8x^4+20x^3+20x^2+86x+94+\frac{140}{x}+\cdots+\frac{20}{x^4} \\
 & \qquad\qquad \underline{8x^4+0x^3+20x^3+20x+40} \\
 & \qquad\qquad\quad 20x^3+0x^2+66x+54+\frac{140}{x}+\frac{50}{x^2}+\cdots \\
 & \qquad\qquad\quad \underline{20x^3+0x^2+50x+50+100/x} \\
 & \qquad\qquad\qquad 16x+4+40/x+50/x^2+90/x^3 \\
 & \qquad\qquad\qquad \underline{16x+0+40/x+40/x^2+80/x^3} \\
 & \qquad\qquad\qquad\quad 4+0/x+10/x^2+10/x^4 \\
 & \qquad\qquad\qquad\quad \underline{4+0/x+10/x^2+10/x^3+20/x^4} \\
 & \qquad\qquad\qquad\qquad \text{remainder: } -20/x^4
\end{array}
$$

11 Prove by induction on k that $(\forall n)\ n\sum_{i=1}^n i^k = \sum_{i=1}^n i^{k+1} + \sum_{p=1}^{n-1}(\sum_{i=1}^p i^k)$.
When $k=0$ we must show that $n\sum_1^n 1 = n^2$ equals $\sum_1^n i + \sum_1^{n-1}(\sum_1^p 1) = \sum_1^n i + \sum_{p=1}^{n-1} p$, which equals $n(n+1)/2+(n-1)n/2 = n/2\{(n+1)+(n-1)\} = n^2$. Thus the formula holds when $k=0$.
Suppose it holds for $k-1$. To show it holds for k we proceed by induction on n. When $n=1$ the formula holds because $1^k = 1^{k+1}+0$. Assuming it holds for $n-1$, and we set $S_n = \sum_{i=1}^n i^k$, we have $S_n = n^k + S_{n-1}$ and hence

$$nS_n = n\,n^k + (n-1)S_{n-1} + S_{n-1} = n^{k+1} + \left[\sum_{i=1}^{n-1} i^{k+1} + \sum_{p=1}^{n-2} S_p\right] + S_{n-1}.$$

The right side is $\sum_{i=1}^n i^{k+1} + \sum_{p=1}^{n-1} s_p$, as required. Thus the inductive hypothesis holds, and the result is true for all n and k.

13 If $xy=d$ and $y^2+dx-db=0$, then we can substitute $y=d/x$ into the second equation to get $d^2/x^2+dx-db=0$. Multiply by x^2/d to get $0=d+x^3-bx^2$ or $x^3+d=bx^2$.

19 We have a spherical triangle with vertices A (north pole), B (New York) and C (London), and we want the distance Ra from B to C. The angle A is the difference in longitudes, or 74°, and we are given that the side c is 59° (90 – 41) and the side b is 48° (90 – 52). By the law of cosines,

$$\cos(a) = \cos(b)\cos(c) + \sin(b)\sin(c)\cos(A) = (.656)(.788) + (.754)(.616)(.276) = .6451$$

which gives $a = 49.8268°$ and distance $Ra = (3936)(49.8268) = 3460$ miles.

23 We are given a spherical triangle with sides $a = 60°$, $b = 75°$, $c = 31°$. The law of cosines gives $.5 = (.2588)(.8572) + (.9659)(.515)\cos(A)$, or $\cos(A) = .5592$. This yields $A = 56°$. Similarly, rotating the letters around, we get $\cos(B)$ and $\cos(C)$ and hence $B = 112°23'$, $C = 29°32'$.

Essay: how did al-Kwarizmi's text differ from its Babylonian sources?

One difference is that al-Kwarizmi stated problems – and methods for their solution – in a general way (using *xay* for a variable). The Babylonians merely provided recipes in specific cases that their students were expected to internalize on their own. For example, al-Kwarizmi would say "take half the number of roots, which in the present instance is 5" while the Babylonians woud have said "take half of 10, which is 5."
Another difference is that al-Kwarizmi provided (geoemtric) proofs that his methods (or algorithms) would provide the correct answers. The idea that proofs were required came from the Greek mathematical school, and was completely absent in Mesopotamia.
A third difference is that al-Kwarizmi classified problems by type, emphasizing the similarity of the solution methods, while no such grouping has been (yet) found in the Babylonian tablets.