Leonhard Euler (1707-1783)

Born in Basel, Switzerland, studied under Johann Bernoulli. 1726, got his Ph.D. On propagation of sound.
In 1727, Daniel Bernoulli convinced Catherine I to give Euler a position at University of St. Petersburg. (Peter II) 1731 became chair of the Math Department there. In 1741, left for a more secure position in Berlin (25 years). In 1766, returned to St. Petersburg until death (Catherine the Great).

Euler introduced the concept of a function and was the first to write $f(x)$ to denote the function $f$ applied to the argument $x$. He also introduced the modern notation for $\sin(x), \cos(x)$.

- letter $e$ for the base of the natural log was due to him,
- letter $\Sigma$ for summations
- letter $i$ to denote the imaginary number.
- Combined Newton and Leibniz approaches to Calculus
- The use of the Greek letter $\pi$ to denote the ratio of circumference to radius was also popularized by Euler, although it did not originate with him.

Most famous textbooks:
- *Introductio in analysin infinitum*, (Intro to Analysis of the Infinite) published in 1748,
- *Institutiones calculi differentialis* (Methods of the Differential Calculus) published in 1755

**Branches of Math influenced by Euler:**

- **Analysis.** Use of power series for exponential, arctan(x), sinh(x) and others such as

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = \lim_{n \to \infty} \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \right) = \frac{\pi^2}{6}.
$$

- **Number Theory:** Euler Phi function, distribution of prime numbers
- **Topology and Graph Theory:** Euler characteristic of polyhedra, Seven Bridges of Konigberg
- **Differential Equations and Applied Math:** use of Fourier series, remainder for Maclaurin series
- **Physics:** Sound waves, Euler-Bernoulli Beam equation, orbits of comets
- **Logic:** Euler diagrams to illustrate reasoning by syllogisms
From Figure 4.17, we can make some predictions about the behavior of unforced solutions. If $\omega$ is very large, then the coefficient $a = 1/(2 - \omega^2)$ is small. In this case solutions of the forced equation

$$\frac{d^2y}{dt^2} + 2y = \cos \omega t$$

are very close to solutions of the unforced equation. A large value of $\omega$ means the external force has very short period and large frequency. The above discussion predicts that such an external force with large frequency has only a small effect on solutions. The harmonic oscillator does not have enough time to respond to a push in one direction before the sign of the external force changes. A typical example is shown in Figure 4.18.

![Figure 4.18](image)

**Figure 4.18**
Solution of
$$\frac{d^2y}{dt^2} + 2y = \cos 10t$$
for initial conditions $y(0) = 0.5$, $y'(0) = 0$.

**Qualitative behavior**
Setting $\omega = \sqrt{2}$, the differential equation becomes

$$\frac{d^2y}{dt^2} + 2y = \cos \sqrt{2} t.$$  

We can predict the behavior of solutions of this equation in two ways. First, we can study solutions of the forced equation

$$\frac{d^2y}{dt^2} + 2y = \cos \omega t$$

for $\omega$ very close to $\sqrt{2}$. As pointed out above, the frequency of the beats decreases and the amplitude of the forced response increases as $\omega$ approaches $\sqrt{2}$. A sequence of solutions with $\omega$ approaching $\sqrt{2}$ is shown in Figure 4.20.

![Figure 4.20](image)

**Figure 4.20**
Solutions of $d^2y/dt^2 + 2y = \cos \omega t$ with $y(0) = y'(0) = 0$ for $\omega = 0.5$, $\omega = 1$, $\omega = 1.2$, and $\omega = \sqrt{2}$.
12.1 Differential Equations and Trig

Euler (1739)

1. Driven harmonic oscillator

\[ 2a \frac{d^2x}{dt^2} + \frac{x}{b} = 0 \quad \text{so} \quad 2a \frac{d^2s}{dt^2} + \frac{s}{b} + \frac{a}{g} \sin \left( \frac{t}{\sqrt{2ab}} \right) = 0 \]

\[ s = C \cos \left( \frac{t}{\sqrt{2ab}} \right) \quad \text{(phase shift omitted)} \]

Postulated \( s = u \cos \left( \frac{t}{\sqrt{2ab}} \right) \), solved for \( u \).

2. Introduction of \( e^x \), \( \sin(x) \), \( \cos(x) \) led Euler to solve

\[ a_n \frac{d^n y}{dx^n} + \ldots + a_2 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = 0 \]

Consider algebraic equation

\[ a_n p^n + \ldots + a_2 p^2 + a_1 p + a_0 = 0 \]

Factor \( p^2 + bx + c \) \( \rightarrow \) \( e^{x} \), \( \sin \left( \frac{\sqrt{4c-b^2}}{2} x \right) \) and \( \cos \left( \frac{\sqrt{4c-b^2}}{2} x \right) \)

3. \( \frac{d^3 y}{dx^3} - y = 0 \) roots of \( a^3 - 1 \) are \( \frac{\sqrt{3}}{2} \), \(-\frac{\sqrt{3}}{2} \), \( \frac{i \sqrt{3}}{2} \)

\[ y = Be^{x/2} + Ce^{-x/2} \sin \left[ \frac{(x+\phi) \sqrt{3}}{2} \right] \quad \text{constants} \ B, C, \phi \]

4. Euler had to explain to Johann Bernoulli, led to

\[ e^{ix} = \cos(x) + i \sin(x) \quad \text{and} \quad \cos(x) = \frac{e^{ix} + e^{-ix}}{2} \]

and finally

\[ e^{\pi i} = -1 \]
12.2 Calculus of Several Variables

1. Interchangeability Theorem (1690s) \[ \frac{\partial}{\partial t} \left[ \int_a^b f(x,t) \, dx \right] = \int_a^b \frac{\partial f}{\partial t} \, dx \] Leibniz

2. Total differential \[ dy = \frac{\partial y}{\partial x} \, dx + \frac{\partial y}{\partial t} \, dt \] Nicolas Bernoulli 1719
   Formula \[ \frac{\partial^2 f}{\partial x \partial t} = \frac{\partial f}{\partial t} \frac{\partial f}{\partial x} \] Alexis Clairaut 1739

3. \[ \iint \frac{dxdy}{x^2+y^2} \rightarrow \int_{\mathbb{R}} \frac{1}{x^2+y^2} \, dx + g(x) \] Euler 1769
   - Surface area
   - Volume of a solid
   - Area
   \[ \int_{\Omega} \int_{\mathbb{R}} dxdy \]

4. PDEs & Wave Equation for vibrating string of length L
   \[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \]
   Daniel Bernoulli 1753
   \[ y(x,t) = \sum a_n \sin \left( \frac{n \pi x}{L} \right) \cos \left( \frac{n \pi t}{L} \right) \] [Fourier 1807]

5. What kind of functions are acceptable as solutions?
Jean Le Rond d'Alembert (1717-1783)

Son of an artillery officer and a nun, abandoned on the steps of the church of St Jean Le Rond. d'Alembert was turned off to the study of theology in College. In 1735-1741, careers as lawyer and medical doctor.

He believed mechanics to be based on metaphysical principles and not on experimental evidence. In 1744, he gave an alternative approach to fluid flow. In 1747, wrote about vibrating strings. The first appearance of the wave equation in print but suffers from the defect that he used mathematically pleasing simplifications of certain boundary conditions which led to results which were at odds with observation. In 1746, started writing an Encyclopédie with Diderot. In 1751, dispute with Euler & stopped publishing papers.

In the 1754 article *Différentiel* in volume 4 of *Encyclopédie*, he suggested that the theory of limits be put on a firm foundation. He was one of the first to understand the importance of functions and, in this article, he defined the derivative of a function as the limit of a quotient of increments of slopes of secant lines.

**Definition (d'Alembert):** One magnitude is said to be the limit of another when the second may approach the first within any given magnitude, however small.

Joseph Louis Lagrange (1736-1813)

Born in Torino (now Italy) to a wealthy family. Starting in 1754, he developed the *Calculus of variations*. In 1766, recommended by Euler and D'Alembert, Lagrange succeeded Euler as the director of mathematics at the Prussian Academy of Sciences in Berlin. In 1772, discovered *Lagrangian points* for 3-body problem. In 1773 proved Wilson's Theorem: \((n-1)! + 1\) divisible by \(p\). In 1786, Frederick the Great died, and Lagrange, gladly accepted the offer of Louis XVI to migrate to Paris. In 1793, asked to head the committee on weights & measures which in 1799 accepted the decimal system. Between 1772 and 1788, Lagrange re-formulated Newtonian mechanics to simplify formulas and ease calculations. These mechanics are called *Lagrangian mechanics*.

Major book (1778): *Mécanique analytique*
Pierre-Simon Laplace (1749-1827)

 Came from a prosperous farming family in Normandy (French coast)
 Age 19, quit school and went to Paris; given a position as
 mathematics professor at the École Militaire
 1770-1773 wrote 13 papers, differential equations and probability
 theory, finally got a job in Paris with the Académie des Sciences
 In 1784 Laplace passed a student named Napoolean Bonaparte
 In 1790, part of committee to establish metric system
 In 1793-95, fled Paris during Reign of Terror
 In 1795, published A philosophical essay on probabilities
 In 1799 published the Traité de Mécanique Céleste
 In 1799, after only six weeks, Napoleon removed Laplace from the
 office of Minister of the Interior, which he held:-
 ... because he brought the spirit of the infinitely small into
 the government.

Adrien-Marie Legendre (1752-1833)

 In 1770, at the age of 18, Legendre defended his (honors) thesis in mathematics and physics
 From 1775 to 1780 he taught with Laplace at École Militaire

 In 1783, He gave a proof of a result due to Maclaurin, that the attractions at an external point lying on
 the principal axis of two confocal ellipsoids was proportional to their masses. He then introduced what
 we call today the Legendre functions and used these to determine, using power series, the attraction of
 an ellipsoid at any exterior point.
 In 1785, wrote a paper on number theory containing “Gauss”
 quadratic reciprocity – Legendre’s proof was unsatisfactory.
 In 1790, part of committee to establish metric system
 In 1793-95, fled Paris during Reign of Terror

 In 1806, developed the least squares method to fit data
 In 1811/1817 published Exercices du Calcul Intégral
 Elliptic integrals, Gamma functions
 1800-1830 tried and failed to PROVE the parallel axiom