The Lebesgue number of a covering

Definition: Let $\mathcal{A}$ be an open covering of a metric space X. A *Lebesgue number* for $\mathcal{A}$ is a positive real number δ such that every subset C of X with diameter less than δ satisfies $C \subset A$ for some $A \in \mathcal{A}$.

Theorem: *Every open covering of a compact metric space (X, d) has a Lebesgue number.*

Proof: Let $\mathcal{A}$ be an open covering of X. Each $x \in X$ belongs to some open set $A_x \in \mathcal{A}$, so there exists a $\delta_x > 0$ such that $B_d(x, \delta_x) \subset A_x$. The family $\{ B_d(x, \delta_x/2) \mid x \in X \}$ is an open covering of X; let $\{ B_d(x_i, \delta_{x_i}/2) \mid i = 1, \ldots, n \}$ be a finite subcovering, and define

$$\delta = \min_{1 \leq i \leq n} \frac{\delta_{x_i}}{2}.$$

We verify that δ is a Lebesgue number for $\mathcal{A}$. Let $C \subset X$ be a set with diameter less than δ, and chooose $x \in C$; then $x \in B_d(x_i, \delta_{x_i}/2)$ for some i, so that for any $y \in C$,

$$d(y, x_i) \leq d(y, x) + d(x, x_i) < \delta + \frac{\delta_{x_i}}{2} \leq \delta_{x_i}.$$

Thus $C \subset B_d(x_i, \delta_{x_i}/2) \subset \mathcal{A}_{x_i}$. ∎