PRACTICE COMBINATORIAL THEORY QUESTIONS

Question 1. Suppose that on a street with 20 houses, 7 have a person ill with the flu.

(a) In how many ways can the presence or absence of flu occur so that all of these 7 houses are next to each other?

(b) In how many ways can the presence or absence of flu occur so that none of these 7 houses are next to each other?

The answers are:

(a) 14

(b) $2 \times \left(\frac{13!}{7!6!}\right)$

Question 2. Solve the recurrence

$$a_{k+1} = 5a_k + 7^k, \qquad k \geq 1$$
$$a_1 = 6.$$

The answer is:

$$a_k = \frac{5^k + 7^k}{2}$$

Question 3. Solve the recurrence

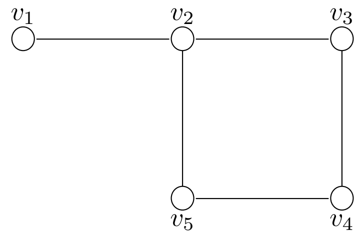
$$a_{n+1} = (n+1)a_n + 7, \qquad n \geq 0$$
$$a_0 = 7.$$

The answer is:

$$a_n = 7n!\,(1 + 1/1! + 1/2! + \cdots + 1/n!)$$

Question 4.

(a) Use the principle of inclusion/exclusion to compute the chromatic polynomial of the following graph.



(b) Find the number of distinguishable ways in which the letters a, a, b, b, c, c, d, d, can be arranged so that two letters of the same kind never appear consecutively.

c) Find the number of distinguishable ways in which the letters a, a, b, b, c, c, d, d, d can be arranged so that there are at least two consecutive letters of type d.

The answers are:

(a) $x^5 - 5x^4 + 10x^3 - 9x^2 + 3x$

(b) $P(8; 2, 2, 2, 2) - 4P(7; 1, 2, 2, 2) + 6P(6; 1, 1, 2, 2) - 4P(5; 1, 1, 1, 2) + P(4; 1, 1, 1, 1)$

(c) $2P(8 : 2, 2, 2, 2) - P(7 : 2, 2, 2, 1)$

Question 5.

(a) Define the chromatic number of the finite graph G.

(b) Show that $p(x) = x^5 - 9x^4 + 22x^3 - 18x^2 + 4x$ is *not* the chromatic polynomial of any graph.

(c) Find a graph G such that

$$P(G, x) = x^7 - 5x^6 + 9x^5 - 7x^4 + 2x^3.$$

Question 6. A word from the alphabet $\{0, 1, 2, 3\}$ is *legitimate* if no two 0's appear consecutively.

(a) Find a recurrence for the number b_n of legitimate words of length n.

(b) Solve the recurrence.

(c) A word from the alphabet $\{0, 1, 2, 3\}$ is *valid* if

   (i) 0 and 2 occur an even number of times, and

   (ii) 1 and 3 occur an odd number of times.

   Find the number c_n of valid words of length n.

The answers are:

(a) $b_{n+1} = 3b_n + 3b_{n-1}$, where $b_1 = 4$ and $b_2 = 15$.

(b) The solution of the recuurence is

$$b_n = \left(\frac{\sqrt{21}+5}{2\sqrt{21}}\right)\left(\frac{3+\sqrt{21}}{2}\right)^n + \left(\frac{\sqrt{21}-5}{2\sqrt{21}}\right)\left(\frac{3-\sqrt{21}}{2}\right)^n$$

(c) For $n \geq 1$, the required number is

$$c_n = \frac{4^n + (-4)^n}{16}$$

Question 7.

(a) Give the definition of a Latin square of order n.

(b) Give the definition of when two Latin squares of order n are orthogonal.

(c) Construct a complete family of orthogonal Latin squares of order 5.

(d) Give an example of a pair of orthogonal Latin squares of order 5, both of which have first row

$$1 \quad 5 \quad 3 \quad 4 \quad 2.$$