

Math 640:454:01 — Fall 2007  
MW5: 3:20 - 4:40 PM HLL-009  
Prof. Bumby

## Chain Partitions

**Statement of the problem:** An exercise asks for a proof of the following theorem of Dilworth:

**Theorem.** *If a partially ordered set has width  $k$ , then it can be written as a disjoint union of  $k$  chains.*

In the text, a proof is given of a result with a similar statement that a partially ordered set with height  $k$  is a union of  $k + 1$  antichains (the number is  $k + 1$  because the height of partially ordered set is defined to be the number of links in the longest chain, and there is one more node in the chain than the number of links). The introduction to Dilworth's Theorem claims that it has a similar proof. This claim is misleading. Most student attempts at a proof sought a natural chain whose removal would decrease the width, but were incomplete. Of course, the theorem says that such chains must exist, but there does not seem to be an easy way to find one directly.

A standard approach is known as the **greedy method**: identify the largest object of the desired type and remove it to leave a simpler problem. One expects that this would apply here, but a proof that the width is decreased is elusive. Moreover, the cases in which it is easiest to show that the width is decreased involve the removal of a **short** chain. The case of an isolated point, incomparable to all others, is an extreme example. This suggests that a different method will be required. We will return to this later.

**The origin of theorem** The paper, "A decomposition theorem for partially ordered sets" by R. P. Dilworth appeared in the journal, *Annals of Mathematics*, vol. 51 (1950), 161–166. The original proof was essentially one induction argument inside another. The proof is organized as an induction on the width of the poset. The basis for this induction is the observation that, in a partially ordered set of width 1, there are no incomparable pairs. This requires the set to be totally ordered, so it is a single chain. In the inductive step of this proof, we have a poset of width  $k$  and we assume that sets of width  $k - 1$  can be written as a disjoint union of  $k - 1$  chains. This will be proved by induction on the number of points in the set. The basis for this induction will be a set of  $k$  points, which must be mutually incomparable if the width is  $k$ . These points are treated as chains of length 1 to establish the desired conclusion. In Dilworth's proof, points are added to this set one at a time, and the decomposition into chains is modified after each point is added. We will give a modification in which the induction is organized differently to allow a simpler proof.

Although the finiteness of the set is used in the proof, in the form of an induction argument, the theorem applies to infinite posets of finite width. This was part of the original paper. The proof uses a "compactness" argument that had appeared in a few places earlier, but has subsequently become a standard tool in logic, combinatorics, and topology.

**Later developments** Other proofs were given later, by Dilworth and others, and the result was related to other major themes in mathematics. A convenient source for Dilworth's papers, along with commentary on work inspired by them, is in "The Dilworth Papers: Selected papers of Robert P. Dilworth", Birkhauser, 1990, ISBN 0-8176-3434-7, which can be found in the library at call number QA171.5.D55 1990. Much of the commentary here is taken from a survey article by K. Bogart, C. Greene, and J. Kung in that volume.

Curiously, the related result on decomposing into antichains, used in the text as a model for this theorem, is not mentioned in Dilworth's paper. Bogart, Greene and Kung do mention it, but they refer to its proof as "trivial". However, they indicate that it can be used as part of a proof of Dilworth's theorem from a difficult theorem in graph theory.

From the beginning, the theorem has been related to classical theorems on "matching". It was later seen to be provable using the duality principle in linear programming. The connection with linear programming allows efficient solutions of other problems to be transformed into efficient decompositions of posets. It is also noted the theorem of Erdős and Szekeres (mentioned at the end of Chapter 2) that a sequence of numbers of length  $n^2 + 1$  contains either an increasing subsequence of length  $n + 1$  or a decreasing subsequence of length  $n + 1$  is a consequence of a decomposition of posets theorem (actually of both forms of the theorem). The sequence  $a_0, a_1, a_2, \dots, x_{n^2}$  is used to build a poset with nodes  $x_i$  for  $0 \leq i \leq n^2$  where  $x_i < x_j$  if  $i < j$  and  $a_i < a_j$ . The increasing subsequences are the chains in this ordering, and the decreasing sequence are the antichains. Small height (respectively, width) gives decomposition into a small number of antichains (respectively, chains) that cannot all be small because the set is large.

**An easily described proof** An attractive proof was given in M. A. Perles, "A proof of Dilworth's decomposition theorem for partially ordered set", Israel J. Math. **1** (1963), 105–107. The change is in how induction on the number of elements is used. The step by step approach is replaced by what is sometimes called the "second principle of induction" in which the desired conclusion is assumed for a set of **any** smaller size. Given a set  $S$  of a particular size and of width  $k$ , two cases are considered. First, if there is an antichain  $P$  in  $S$  with  $k$  elements that is neither the set of maximal elements of  $S$  nor the set of minimal elements of  $S$ , then  $S$  splits into two smaller sets:  $S^+$  is the set of elements greater than or equal to some element of  $P$ ; and  $S^-$  is the set less than or equal to some element of  $P$ . In this case, it is easy to show that the union of these two sets includes all elements of the original set and their intersection is  $P$ . The decompositions of the two subsets are joined at elements of  $P$  to form the required chains. Second, if no such antichain  $P$  can be found, then removing **any** chain joining a maximal element to a minimal element (even one with only one or two elements) leaves a set of smaller width.

**A simplified version of Dilworth's proof** The proof will only be an induction on the size of the set. In the induction step, the statement to be proved is that a set can be decomposed into a number of chains at most equal to its width (a simple argument that fewer chains will not suffice will appear later in the proof). This statement will be assumed for all smaller sets, and the empty set—the only set of size 0—may be taken as the basis for this induction. The actual induction step for a set  $S$  will involve only subsets of  $S$ , which cannot have larger width, so there is still an implicit induction on width.

Given a poset  $S$ , let the width of  $S$  be denoted  $k$ . Choose a **minimal point**  $x$  of  $S$  and consider  $S' = S \setminus \{x\}$ . Since  $S'$  is a proper subset of  $S$ , it has fewer elements and its width is at most  $k$ , so it is a disjoint union of at most  $k$  chains  $C_i$ , with  $1 \leq i \leq k$  (if fewer chains suffice, let  $C_i$  be empty for any unused values of  $i$ ). Write each  $C_i = U_i \cup V_i$ , where  $U_i$  consists of the elements of  $C_i$  that are greater than  $x$  and  $V_i$  consists of those that are incomparable to  $x$ . Since  $x$  was minimal in  $S$ , and not in  $S'$ , these are the only possibilities. It is possible that  $U_i$  or  $V_i$  is empty. Note that  $U_i \cup \{x\}$  is a chain. Now consider  $S_i = S' \setminus U_i$  for each  $i$ . If the width of any  $S_i$  is less than  $k$ , write  $S_i$  as a union of  $k - 1$  disjoint chains. These chains, together with  $U_i \cup \{x\}$  give the required decomposition of  $S$ .

It remains to show that not all  $S_i$  can have width  $k$ . We prove this by contradiction.

First, note that any element greater than an element of  $U_i$  belongs to  $U_i$ ; and, since  $C_i$  is a chain, any element smaller than an element of  $V_i$  belongs to  $V_i$ . For each  $i$ , let  $P_i$  be an antichain of size  $k$  contained in  $S_i$ .

Since an antichain cannot contain two elements in the same  $C_j$ , any antichain of size  $k$  must contain exactly one element in each  $C_j$ , and the element of  $P_i$  belonging to  $C_i$  is in  $V_i$ . Now, for each  $j$ , let  $v_j$  be the smallest element of  $C_j$  belonging to one of the  $P_i$ . Since each  $v_j \in V_j$ , the set consisting of  $x$  and all of the  $v_j$  has size  $k + 1$ , so it cannot be an antichain. Since  $x$  is incomparable to all of the  $v_j$ , there must be indices  $j_0$  and  $j_1$  with  $v_{j_1} > v_{j_0}$ . However, if  $c_{j_1}$  is the element of  $P_{j_0}$  belonging to  $C_{j_1}$ , we have  $c_{j_1} > v_{j_1} > v_{j_0}$ , contradicting the claim that  $P_{j_0}$  is an antichain.

**An operation hidden in the proof** In the article “Some combinatorial problems on partially ordered sets” in R. Bellman and M. Hall, Jr., eds. *Combinatorial Analysis (Proceedings of the Tenth Symposium in Applied Mathematics, Columbia University, 1958)*, American Mathematical Society, 1960, pp. 85-90, Dilworth introduced an order relation on antichains of maximum size in a poset. If  $P$  and  $P'$  are such antichains, one says that  $P \geq P'$  if, for every  $x \in P$ , there is  $x' \in P'$  with  $x \geq x'$ . Several properties of this relation are proved leading to a proof that the poset of all antichains of maximum size with this relation is a **distributive lattice**. The key step is the construction of the **glb** of two antichains that are elements of this lattice as the set of minimal points in the union of the two antichains. This reveals that the contradiction in the proof arose from the set of  $v_i$  being required to be an antichain from the relation of its construction to the lattice operations on antichains of maximum size, while being barred from being an antichain because all of its elements are incomparable with  $x$ .

Another observation in the same article shows:

**Proposition.** *Given a poset  $X$  of width  $k$ , let  $X'$  be the union of all antichains of size  $k$  in  $X$ . If  $C$  is a chain in the decomposition of  $X'$  into  $k$  chains, then  $X \setminus C$  has width less than  $k$ .*

*Proof.* If there is an antichain  $P$  of size  $k$  in  $X \setminus C$ , all of its elements must be in  $X'$ , but this requires that one of those points lie in  $C$ .

This is evidence that, in contrast to the greedy approach, the longest chain should be left for last.