Math 640:454:01 — Fall 2007
MW5: 3:20 - 4:40 PM   HLL-009
Prof. Bumby

Chapter 2, Additional Exercise 13

Statement of the problem: A person wishes to visit six cities, each exactly twice, and never visiting the same city twice in a row. In how many ways can this be done?

Comment: This does not seem to be appropriate for this chapter, although a solution will be given that uses methods that fit with the methods of the chapter. However, techniques of **generating functions** and **inclusion-exclusion** are more natural. We will revisit the problem when those methods are encountered.

Introduction: Without the restriction on consecutive visits, the number of paths (which we will call **tours**) of the six cities is $P(12; 2, 2, 2, 2, 2, 2)$ described in Section 2.11.2, which is equal to the **multinomial coefficient** $C(12; 2, 2, 2, 2, 2, 2)$ by Theorem 2.6. This number is

$$\frac{12!}{(2!)^6} = 7,484,400$$

This number is too large to allow us to list all tours and rule out those with repeated cities. In order to conveniently refer to the desired tours, we will call them **non-repeating**.

Parametrize: Instead of fixing the number of cities at 6, let it be a variable n. Ideally, the problem could be solved to give a simple function of n for the number of non-repeating tours. I was not able to find such a formula by the methods appearing so far in the course, but this approach allows knowledge of the solution for the first several values to be used to find the next. We let T_n denote the **total** number of tours visiting n cities twice, and let S_n be the number of non-repeating tours.

When $n = 1$, $T_1 = 1$ and the one tour repeats the one city, so $S_1 = 0$. When $n = 2$, $T_2 = 6$. All tours can be listed. They are: AABB, ABAB, ABBA, BAAB, BABA, BBAA (using lexicographic order to assure that all are included) and only ABAB and BABA are non-repeating, so $S_2 = 2$.

In general,

$$T_n = n(2n - 1)T_{n-1}$$

so $T_3 = 90$. Solution by enumeration is beginning to look tedious.

The real base case: The case $n = 0$ needs to be included. Because of the convention that $0! = 1$ and $2^0 = 1$, the formula gives $T_0 = 1$. The one tour counted in this way is the **empty tour** and this does not visit one city twice in succession (because it visits no cities), so $S_0 = 1$.

Symmetry: Interchanging the symbols A and B in the list for $n = 2$ reflects the first three pairs into the last three. A tour is non-repeating if and only if its reflection is since we are considering a property in which all cities are treated equally. For a general value of n, each tour can be considered as a representative of $n!$ tours obtained by permuting the **names of the cities** in the tour. All of these tours are distinct. Thus T_n is divisible by $n!$. We write $T_n = n! \cdot U_n$, and note that U_n is an integer. Using the inductive description of T_n, we find that $U_0 = 1$ and $U_n = (2n-1)U_{n-1}$ for $n > 0$.

Furthermore, if a tour is non-repeating, so are all $n!$ tours obtained by permuting names. In other words, we are considering a property of the **equivalence classes** of tours related by permuting names. Any method of listing representatives of all tours that avoids including two equivalent tours will succeed in counting in counting the number Z_n of non-repeating classes, where $S_n = n! \cdot Z_n$.

Now, $U_2 = 3$ and $U_3 = 15$. This makes $n = 3$ accessible to enumeration, but $U_4 = 105$, so enumeration again becomes tedious.

Induction: The formula $U_n = (2n-1)U_{n-1}$ has a simple combinatorial interpretation. After identifying the first city visited, one can ask for the time of the next visit to that city. The number of intermediate stops is a number between 0 and $2n-2$, giving $2n-1$ cases. If the length of this gap is fixed, there are $2n-2$ places to be filled to complete the tour. If these are filled in order by a tour of $n-1$ cities, one obtains a tour of n cities. Changing the name of the first city and permuting the names on the inserted tour gives all permutations of the names on the constructed tour. To count equivalence classes, we may fix the name of the first city to be A.

If the gap between visits to A is zero, the resulting tour is repeating, so none of these tours can be counted. Otherwise, if the inserted tour is non-repeating, so is the combined tour. However, there are other ways that a suitable tour can be found. If the size of the gap is between 1 and $2n-3$, and the same city is visited immediately before and immediately after the second visit to A, and if we choose the name of that city to be B, then we need to fill the gaps between A and BAB with a tour of $n-2$ cities. Since all names are available, every equivalence class has a representative with this property. In selecting these tours to avoid repetitions, we may choose a non-repeating tour of $n-2$ cities or a tour with a single repetition surrounding BAB if there are spaces on both sides of this string. This process may be repeated to get

$$Z_n = (2n-2)Z_{n-1} + (2n-3)Z_{n-2} + (2n-5)Z_{n-3} + \ldots$$

These sums are simple enough to give $Z_3 = 5$ and $S_3 = 30$, $Z_4 = 36$ and $S_4 = 864$, $Z_5 = 329$ and $S_5 = 39,480$, $Z_6 = 3,655$ and $S_6 = 2,631,600$.

A further simplification: Writing out the terms of the sum obtained at the end of the last section reveals that the terms at the end of the sum are **exactly the same** for all values of n. Alternatively, the strings beginning ABAB contribute Z_{n-2} tour classes to Z_n while the tours in which the initial A and the string BAB are separated behave exactly like the two appearances of the first city in a tour of $n-1$ cities. The other $2n-2$ contributions of Z_{n-1} to Z_n are not affected, so we obtain

$$Z_n = (2n-1)Z_{n-1} + Z_{n-2}$$

for $n \geq 2$.