Math 640:454:01 — Fall 2007
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Chapter 2, Additional Exercise 13, part 2

Background: A previous supplement used methods of chapter 2 of the text to study the problem: "A person wishes to visit six cities, each exactly twice, and never visiting the same city twice in a row. In how many ways can this be done?"

In this solution, "six" was replaced by a parameter n, and the solution in terms of n was denoted S_n. The solutions were shown to fall into blocks of size $n!$ obtained by permuting the names of the cities. Thus, it was more convenient to study Z_n defined by $S_n = (n!)Z_n$. This function was shown to be characterized by $Z_0 = 1$, $Z_1 = 0$, and

$$Z_n = (2n-1)Z_{n-1} + Z_{n-2}$$

for $n \geq 2$. We have now reached the point where we have **generating functions** available and have seen how a recurrence can produce a closed form expression for a generating function.

Rate of growth: The Z_n are known to be nonnegative integers, so the recurrence shows that

$$Z_n - (2n-1)Z_{n-1} = Z_{n-2} \geq 0$$

for $n \geq 2$, which implies that $Z_n \geq (2n-1)Z_{n-1}$, so that $Z_n > n!$ for $n \geq 4$. Although one could form the formal series $\sum Z_n x^n$, this series would not converge except for $x = 0$, so that it is unlikely to be represented by a familiar function.

This suggests that we let $Z_n = (n!)X_n$. The X_n are no longer integers, but the recurrence can be used to show that $X_{n+1} < 2X_n$. The series

$$F(x) = \sum X_n x^n = \sum Z_n \frac{x^n}{n!}$$

will converge for $|x| < 1/2$. Note that the **ordinary generating function** of X_n is the same as the **exponential generating function** of Z_n. The use of an exponential generating function here is suggested only to control the growth of the coefficients; the modified **convolution** corresponding to a product of exponential generating functions will not appear.

Consequences of the recurrence: Multiplying the recurrence by $x^{n-2}/(n-2)!$ gives

$$\frac{Z_n x^{n-2}}{(n-2)!} = (2n-1)\frac{Z_{n-1}x^{n-2}}{(n-2)!} + \frac{Z_{n-2}x^{n-2}}{(n-2)!}.$$

This can be rewritten as

$$n(n-1)X_n x^{n-2} = (2n-1)(n-1)X_{n-1}x^{n-2} + X_{n-2}x^{n-2}.$$

These equations are only valid for $n \geq 2$. The next step will be to sum these for those values of n and interpret the result. The sum on the terms on the left is $F''(x)$ and the last term on the right sums to $F(x)$, so it remains to interpret the first term on the right. We will see that it can be expressed as a combination of $(n-1)X_{n-1}x^{n-2}$, that appears in $F'(x)$, and $(n-2)(n-1)X_{n-1}x^{n-3}$, that appears in $F''(x)$. The terms for $n \geq 2$ account for all terms of the series for F, F' and F''. Since $2n-1 = 2(n-2)+3$, we have

$$(2n-1)(n-1)X_{n-1}x^{n-2} = 2x(n-1)(n-2)X_{n-1}x^{n-3} + 3(n-1)X_{n-1}x^{n-2}.$$

Thus,

$$F''(x) = 2xF''(x) + 3F'(x) + F(x).$$

The values of X_0 and X_1 give initial conditions.

Solving the equation: We now have

$$(2x-1)F''(x) + 3F'(x) + F(x) = 0;\ F(0) = 1;\ F'(0) = 0.$$

General properties of differential equations tell us that this **initial value problem** has a unique solution and that the solution can be given by a series that converges for $|x| < 1/2$. In a typical course on Differential Equations, the usual approach to an equation like this is to find a series solution. Of course, we already know how to find as many terms of the series as we want. We hope that a closed form expression for the solution will appear.

However, **Maple** is available and it knows more ways to solve equations than are found in introductory course, so I asked Maple if it could solve the equation. It gave a solution, although it was necessary to modify the expression it gave to get a more useful form. The solution of

$$F(x) = (1-2x)^{-1/2} e^{(1-2x)^{1/2}-1}.$$

I leave it as an exercise to check that this function is a solution of the initial value problem.