

Math 640:454:01 — Fall 2007  
MW5: 3:20 - 4:40 PM HLL-009  
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## Generating Functions

**Ordinary Generating Functions** For a sequence  $(a_k)$ , with the understanding that  $k$  is a nonnegative integer, the expression

$$A(x) = \sum_{k=0}^{\infty} a_k x^k$$

gives a **single object** representing the **entire sequence**. This is particularly useful when the series is the Taylor series of a familiar function. An important example is  $a_k = 1$  for all  $k \geq 0$ . The series  $A(x)$  is a **geometric series** that is known to converge to  $(1 - x)^{-1}$ .

Other examples are obtained by performing simple operations on this one.

When you met infinite series in Calculus, the Taylor series was given star billing. It allowed the coefficients of a series representation of a function to be expressed in terms of the **values of the derivatives** of the function at a single point (which will be  $x = 0$  in all examples in this course). The statement is just what you expect: if you **formally** find  $(d/dx)^n A(x)$  using the series, the terms of degree less than  $n$  become zero, the terms of degree greater than  $n$  retain at least one factor of  $x$ , so they become zero at  $x = 0$  and  $a_n x^n$  becomes the constant  $a_n n!$ , and Taylor's Theorem says nothing more than  $a_n n! = A^{(n)}(0)$ . Because of this, differentiation and integration will play an important role in constructing new series from old, but there are places where a literal use of Taylor's formula is more complicated than other methods. Taylor's formula, with an error term, still plays a role. It shows that the series has the expected limit.

Note that Taylor's formula produces the coefficients of the series using values of successive derivatives at zero; no simpler relation can be expected to give correct answers. In particular, the  $A(k)$  (which were observed in some answers on the second midterm, offer no shortcut to finding the  $a_k$ .

**Substitution** If  $g(x)$ , with  $g(0) = 0$ , is a function having a Taylor series, then the series for  $A(g(x))$  is formed by expanding

$$A(g(x)) = \sum_{k=0}^{\infty} a_k g(x)^k.$$

The requirement that the series for  $g(x)$  **has no constant term** assures that the term in  $x^n$  in the resulting series arises from only **finitely many terms** of the original series.

A simple example is

$$\frac{1}{1-cx} = \sum_{k=0}^{\infty} (cx)^k = \sum_{k=0}^{\infty} c^k x^k.$$

This is the generating function of a sequence that is a geometric progression.

A more impressive example of the power of substitution is

$$\frac{1}{1-x^2} = \sum_{k=0}^{\infty} (x^2)^k = \sum_{k=0}^{\infty} x^{2k}.$$

The coefficient of  $x^n$  in this series is 0 if  $n$  is odd and 1 if  $n$  is even. It is much easier to obtain this result by substitution than by applying Taylor's formula to  $(1-x^2)^{-1}$ .

**Linearity** The operation producing the generating function  $A(x)$ , and its inverse, are **linear operators**. Given sequences  $(a_k)$  and  $(b_k)$ , the sequence  $\alpha(a_k) + \beta(b_k)$  is the sequence  $(\alpha a_k + \beta b_k)$ , whose generating function can be written as  $\alpha A(x) + \beta B(x)$  and whose value at each  $x$  is the corresponding **linear combination** of  $A(x)$  and  $B(x)$ . This allows the **method of partial fractions** to be used to identify sequences from their generating function when the generating function is a quotient of polynomials (known as a **rational function**). This gives a second computation of the coefficients of  $(1-x^2)^{-1}$ :

$$\frac{1}{1-x^2} = \frac{1}{2} \left( \frac{1}{1+x} + \frac{1}{1-x} \right) = \sum_{k=0}^{\infty} \frac{1}{2} (1 + (-1)^k) x^k.$$

Of course, **all methods must give the same answer**.

**Multiplication and convolution** If the usual **long multiplication** method is employed to find the coefficients in  $C(x) = A(x)B(x)$ , we get

$$c_n = \sum_{k=0}^n a_k b_{n-k}.$$

This operation on series is called **convolution**. This formula is often not useful for computing the  $c_n$ : its chief value is in recognizing that, for combinatorial reasons, one sequence is the convolution of a particular pair of sequences. The convolution formula expresses that the  $n^{\text{th}}$  term of a sequence counts the union of events, indexed by  $k$  between 0 and  $n$ , each of which calls for the satisfying the property counted by  $a_k$  and the property counted by  $b_{n-k}$ . Often, this allows an identity satisfied by a generating function to be found by a combinatorial argument. Several examples are given in the text, but not until section 6.4. The examples are needed earlier to help explain the role of

convolution. Exercise 15 of Section 5.4 is an example of this use of the convolution formula. The operation  $S$  taking the sequence  $(b_n)$  to the sequence whose general term is

$$(Sb)_n = \sum_{k=0}^n b_k$$

is simply convolution with the sequence  $(a_n)$  where  $a_n = 1$  for all  $n$ . The notation used in this exercise may have contributed to the confusion; here we attempt to emphasize that  $S$  is intended to be an operation taking one sequence to another. At the level of generating functions, the operation of  $S$  is multiplication by  $(1-x)^{-1}$ . Associativity of multiplication implies associativity of convolution, and the action of  $S^p$  is multiplication by  $(1-x)^{-1}$   $p$  times, which can be described more simply as multiplication by  $(1-x)^{-p}$ . The formula to be proved in this exercise uses the **binomial theorem** (which explains why the exercise appears in Section 5.4) to write a formula for the coefficients of  $(1-x)^{-p}$  and express the result as a convolution with this sequence.

On the second midterm, the operator  $S$  was to be applied to the **Fibonacci sequence** whose generating function is  $x(1-x-x^2)^{-1}$ . All that is needed in the way of an answer is  $x(1-x)^{-1}(1-x-x^2)^{-1}$ . The poor response revealed that the exercise on the operator  $S$  only served to convince many of you that, “You are not expected to understand this” (a famous quote from a computer manual). I hope that I convinced you that the only difficulty is finding a suitable notation to describe the action of  $S$  on a sequence.

**More examples** Section 6.4.2 works through the determination of generating function of the **Catalan numbers**  $C_n$ . These numbers describe the number of ways to insert parentheses so that a product of  $n$  terms is realized as a sequence of steps each of which multiplies only **two** sub-expressions that are either single terms or parenthesized expressions. A combinatorial argument (based on the treatment in “Concrete Mathematics”) leading directly to a simple formula for the Catalan numbers was mentioned in lecture (somewhere in our discussion of chapter 2). Page 385 of the text shows how this formula may be obtained from the binomial theorem from the generating function (obtained as the solution of a different problem in section 6.4.1).

The convention for the trivial cases is that  $C_0 = 0$  since no expression exists, and  $C_1 = 1$  since there is always exactly one empty set of parentheses. For  $n \geq 2$ , there is always a **last** multiplication, that combines a product of  $k$  terms with a product of  $n-k$  terms, where  $0 < k < n$ . By requiring  $C_0 = 0$ , we have assured that the description of the convolution may be extended to a sum from  $k = 0$  to  $k = n$  without changing the result when  $n \geq 2$ . However, when  $n = 1$  this product is zero, but our definition of  $C_1$ , necessary for those cases in which one of the factors in the last multiplication is a single variable, requires  $C_1 = 1$ . If

$$F(x) = \sum_{n=0}^{\infty} C_n x^n,$$

then  $F(x) \cdot F(x)$  gives a function whose  $n^{\text{th}}$  coefficient is the convolution  $\sum C_k C_{n-k}$ . This agrees with  $C_n$  except when  $n = 1$  where it is 0 while  $C_1 = 1$ . In other words,

$$(F(x))^2 = F(x) - x.$$

The quadratic formula gives

$$F(x) = \frac{1 \pm \sqrt{1 - 4x}}{2}$$

Here, the + sign must be rejected because it leads to  $F(0) = 1$ , while we need  $F(0) = 0$ .

I recently encountered a paper in which the author needed to know the number  $r_n$  of rational functions  $P(x)/Q(x)$  with coefficients in the integers modulo  $p$  (for prime  $p$ ) with  $\deg Q(x) = n$ ,  $\deg P(x) < n$ , and  $\gcd(P(x), Q(x)) = 1$ . By declaring that we are counting rational functions, the operation of multiplying both  $P(x)$  and  $Q(x)$  by the same nonzero constant has no effect on the fraction. This allows us to assume that the leading coefficient of  $Q(x)$  is 1, so there are  $2n$  coefficients to be determined to fix  $P(x)/Q(x)$ . However, not all of them may be in lowest terms. In addition to counting reduced fractions, we may also consider a second problem of counting **all** pairs  $(P(x), Q(x))$  with the requirement on degree and  $Q(x)$  having leading coefficient 1, but no requirement on  $\gcd(P(x), Q(x))$ . The number of such pairs is  $s_n = p^{2n}$ . The two counts are related by identifying the degree of  $\gcd(P(x), Q(x))$ . By expressing a pair of polynomials as a fraction with a common factor of degree  $k$  in both numerator and denominator, we get

$$p^{2n} = s_n = \sum_{k=0}^n p^k r_{n-k}$$

where the factor  $p^k$  counts the number of polynomials of degree  $k$  normalized to have leading coefficient 1. If  $S(x)$  denotes the generating function of  $s_n$  and  $R(x)$  denotes the generating functions of  $r_n$ , we have  $S(x) = (1 - p^2x)^{-1} = R(x) \cdot (1 - px)^{-1}$ . Thus,  $R(x) = (1 - px) \cdot (1 - p^2x)^{-1}$ . The expansion of this convolution contains only one term when  $n = 0$  and two terms for  $n > 0$ , so we see that  $r_0 = 1$  and  $r_n = p^{2n} - p^{2n-1}$  for  $n > 0$ .

There is a second way to get this result. This is encouraging because you are more confident of your answer if you have more than one way to deal with the subtle points of combinatorial analysis. The second approach is to divide  $Q(x)$  by  $P(x)$ . If  $\deg P(x) = k < n$ , then the quotient will be an arbitrary polynomial of exact degree  $n - k$ . There are  $p - 1$  choices for the leading coefficient since it must not be zero, and  $p$  choices for the remaining coefficients. Thus, there are  $(p - 1)p^{n-k}$  choices for the quotient for all  $k < n$ . Subtracting this quotient leaves a fraction  $R(x)/P(x)$  with denominator of degree  $k$  that will be in lowest terms if  $P(x)/Q(x)$  is. The formula given by this construction is a convolution equation

$$r_n = \sum_{k=0}^{n-1} (p - 1)r_k p^{n-k}.$$

This is only valid when there is an actual quotient, i.e., for  $n > 0$ . For  $n = 0$ ,  $r_0 = 1$ , but the sum is an empty sum having value zero. Thus,

$$R(x) = 1 + R(x) \cdot (p - 1) \left( \frac{1}{1 - px} - 1 \right) = 1 + R(x) \left( \frac{p(p - 1)x}{1 - px} \right)$$

Thus,

$$\frac{1 - p^2x}{1 - px}R(x) = 1,$$

in agreement with our previous result.

The step of inverting the proper fraction  $R(x)/P(x)$  and dividing, as we did with original fraction  $P(x)/Q(x)$ , can be repeated to implement the computation of  $\gcd(P(x), Q(x))$  by the **Euclidean Algorithm**. Only those fractions in lowest terms will avoid a numerator of zero except when the denominator is a nonzero constant.

**Stirling Numbers** The Stirling numbers of the second kind were introduced in connection with “Occupancy Problems” in Section 2.10. Exercise 22 of that section asks for a **combinatorial argument** to show that

$$S(n, k) = kS(n - 1, k) + S(n - 1, k - 1).$$

if  $(n, k) \neq (0, 0)$ . This is easy! The Stirling number  $S(n, k)$  counts the number of ways of putting  $n$  **distinguished balls** in  $k$  **undistinguished cells** with **no cell empty**. Since the desired recurrence compares two consecutive values of  $n$ , which is the number of **distinguished** balls, we need only consider the effect of placing the **last** ball. Either the first  $n - 1$  balls fail to assign a ball to every cell, which means that the last ball must be assigned to an empty cell, and that empty cell is the **only** empty cell, giving  $S(n - 1, k - 1)$  cases; or the  $n - 1$  balls already use all  $k$  cells, which happens in  $S(n - 1, k)$  ways, and the last ball may be assigned to any of the  $k$  cells. We handle the vacuous case by **defining**  $S(0, 0) = 1$ . In cases where  $k < 0$  or  $k > n$ ,  $S(n, k) = 0$  and the recurrence computes  $S(n, 0) = 0$  if  $n > 0$ , as one would expect.

This recurrence is exactly the sort of expression that can be used to find a generating function. Indeed, in “generatingfunctionology”, H. S. Wilf uses it to find **three** generating functions:

$$\sum_k S(n, k)y^k, \quad \sum_n S(n, k)x^n, \quad \text{and} \quad \sum_{n,k} S(n, k)x^n y^k.$$

We describe only the second of these, which we call  $B_k(x)$ , here. Our observations about  $S(n, 0)$  tell us that  $B_0(x) = 1$ . For  $k > 0$ , multiplying the recurrence by  $x^n$  and summing gives

$$B_k(x) = xB_{k-1}(x) + kB_k(x).$$

Solving this for  $B_k(x)$  gives

$$B_k(x) = \frac{x}{1 - kx}B_{k-1}(x)$$

for  $k > 0$ . This is a recurrence giving each  $B_k$  in terms of the previous one that is easily turned into a single expression

$$B_k(x) = \frac{x^k}{(1 - x)(1 - 2x) \cdots (1 - kx)}.$$

To identify the ultimate expansion in powers of  $x$ , the terms of this expression are expanded in partial fractions. I **leave it as an exercise** to show that  $1/(1 - rx)$  is multiplied by

$$(-1)^{k-r} \frac{r^{k-1}}{(r-1)!(k-r)!}$$

and hence that

$$S(n, k) = \sum_{r=1}^k (-1)^{k-r} \frac{r^n}{r!(k-r)!},$$

which is essentially formula (2.8) of the text. The text gives a different proof of this formula in Section 5.5.3 using an **exponential generating function**. Although the two methods lead to the same formula, they do not appear to have much in common.

**Exponential Generating Functions** Instead of multiplying  $a_n$  by  $x^n$ , as in the ordinary generating, one can form an **exponential generating function** of the sequence  $(a_n)$  by the formula

$$\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}.$$

This time, if  $a_n = 1$  for all  $n$ , the generating function is  $e^x$ . There are two reasons for considering exponential generating functions: (the analytic reason) the  $a_n$  may grow so rapidly that the ordinary generating function doesn't converge except at  $x = 0$ ; (the combinatorial reason) the  $a_n$  count **permutations**, which require a different form of convolution when counting problems are analyzed. The analytic reason was used when we determined the generating function for "non-repeating tours" arising from "Additional Problem 13" from chapter 2. The combinatorial reason was explored in Section 5.5. In particular, Example 5.33 showed how to count RNA chains satisfying additional restrictions.

A major difference between these two types of generating functions is in the formula for a product of two series. This formula leads to a different convolution of series that are used to determine exponential generating functions. Although the series are multiplied as before, if  $C(x) = A(x)B(x)$  and  $A(x)$ ,  $B(x)$ , and  $C(x)$  are exponential generating functions, then

$$\frac{c_n}{n!} = \sum_{k=0}^n \frac{a_k}{k!} \frac{b_{n-k}}{(n-k)!}.$$

To get the convolution formula, multiply by  $n!$  and recognize the factor independent of of the coefficients of  $A(x)$  and  $B(x)$  to get

$$c_n = \sum_{k=0}^n \binom{n}{k} a_k b_{n-k}.$$

This has the effect of counting each object formed by the  $k^{\text{th}}$  term of  $A(x)$  and the  $(n - k)^{\text{th}}$  term of  $B(x)$  as many times as there are building a string of length  $n$  from string of length  $k$  and  $(n - k)$ .

The quantities  $k!S(n, k)$  solve the **Occupancy problem** of placing  $n$  **distinguished balls** in  $k$  **distinguished cells** with **no cell empty**. The act of distinguishing the cells simply introduces the factor of  $k!$ . However, the nature of the question suggests a different analysis that determines the exponential generating function. In particular, we can ask how the last **cell** is filled. This distinguishes a certain subset of the string on  $n$  distinguished balls, and if we let  $r$  denote the size of this subset, the desired quantity is the sum for all  $r$  of the number of subsets of size  $r$  times the number of ways of assigning these  $r$  balls to the last cell times the number of ways of assigning the remaining  $n - r$  balls to the previous  $n - 1$  cells. The first of these numbers is the binomial coefficient appearing in the convolution formula for exponential generating functions and the other two numbers are the solutions of the corresponding occupancy problem with  $(n, k)$  replaced by  $(r, 1)$  and  $(n - r, k - 1)$ . Thus, the exponential generating function

$$G_k(x) = \sum_{n=0}^{\infty} k!S(n, k) \frac{x^n}{n!}$$

is the product of  $G_1(x)$  and  $G_{k-1}(x)$ . Once  $G_1(x)$  is known, this gives  $G_k(x) = G_1(x)^k$ . Our previous determination of the generating function  $B_1(x) = x/(1 - x)$  shows that  $S(0, 1) = 0$  and  $S(n, 1) = 1$  for  $n > 0$ . Multiplying these numbers by  $1! = 1$  and using them as coefficients of an **exponential** generating function, shows the  $G_1(x) = e^x - 1$ . This was the starting point of Section 5.5.3. The extra details show the role of the convolution formula in motivating the decision to use an exponential generating function.