SUPPLEMENTARY QUESTIONS 1

The following exercises show that if $C \subset \mathbb{R}$ is an uncountable closed subset, then $C \sim \mathbb{R}$.

Definition 1. A nonempty subset $P \subseteq \mathbb{R}$ is said to be *perfect* iff the following conditions are satisfied:

(i) P is closed.

(ii) If $x \in P$ and I is an open interval such that $x \in I$, then there exists a point $y \in I \cap (P \smallsetminus \{x\})$.

Definition 2. Suppose that $X \subseteq \mathbb{R}$ and that $x \in X$. Then x is a *condensation point* of X iff for every open interval I such that $x \in I$, the set $I \cap X$ is uncountable.

Question 1. Let $C \subseteq \mathbb{R}$ is an uncountable closed subset and let

$$D = \{x \in C \mid x \text{ is } \textit{not} \text{ a condensation point of } C\}.$$

Prove that:

(a) D is a countable set.

(b) $C \smallsetminus D$ is a perfect subset of $\mathbb{R}$.

Question 2. Prove that if $P \subseteq \mathbb{R}$ is perfect, then $P \sim \mathbb{R}$.