PRACTICE FIRST MID-TERM QUESTIONS

Question 1. Prove that if A is any set, then $A \not\sim \mathcal{P}(A)$.

Question 2. (a) State the Cantor-Bernstein Theorem.

(b) Prove that $\mathbb{N} \sim \mathbb{Z}[1/2]$, where $\mathbb{Z}[1/2] = \{m/2^n \mid m \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$.

(c) Let $\mathrm{Surj}(\mathbb{N})$ be the set of surjective functions $f : \mathbb{N} \to \mathbb{N}$. Prove that $\mathcal{P}(\mathbb{N}) \sim \mathrm{Surj}(\mathbb{N})$.

Question 3. (a) Give the definition of a linear ordering.

(b) Give the definition of an isomorphism between two linear orders $\langle L_1, < \rangle$ and $\langle L_2, \prec \rangle$.

Prove or disprove each of the following statements.

(c) $\langle \mathbb{Q} \smallsetminus (0,1), < \rangle \simeq \langle \mathbb{Q} \smallsetminus [2,3], < \rangle$.

(d) $\langle \mathbb{Q} \smallsetminus \mathbb{Z}, < \rangle \simeq \langle \mathbb{Q}, < \rangle$.

(e) $\langle \mathbb{Z}[1/2], < \rangle \simeq \langle \mathbb{R} \smallsetminus \mathbb{N}, < \rangle$, where $\mathbb{Z}[1/2] = \{m/2^n \mid m \in \mathbb{Z} \text{ and } n \in \mathbb{N}\}$.

Question 4. Determine whether each of the following wffs is a tautology.

(a) $(A \to (B \to (A \leftrightarrow B)))$

(b) $((P \wedge Q) \to (P \to Q))$

Question 5. Suppose that α is a wff which only involves the connectives $\{ \wedge, \vee \}$ and the sentence symbols $\{ A_1, \cdots, A_n \}$. Prove that if ν is a truth assignment such that

$$\nu(A_1) = \nu(A_2) = \cdots = \nu(A_n) = T,$$

then ν satisfies α.

(*Hint:* Argue by induction on the length of the wff α.)

Question 6. (a) State the Compactness Theorem for propositional logic.

(b) Let $\{S_n \mid n \in \mathbb{N}\}$ be a collection of finite subsets of $\mathbb{N}$ such that for each finite subset $F \subset \mathbb{N}$, there exists a subset $A_F \subseteq \mathbb{N}$ with $|A_F \cap S_n| = 1$ for all $n \in F$. Prove that there exists a subset $A \subseteq \mathbb{N}$ such that $|A \cap S_n| = 1$ for all $n \in \mathbb{N}$.

(c) Give an example of a collection $\{T_n \mid n \in \mathbb{N}\}$ of subsets of $\mathbb{N}$ satisfying both of the following conditions.

(i) For each finite subset $F \subset \mathbb{N}$, there exists a subset $B_F \subseteq \mathbb{N}$ such that $|B_F \cap T_n| = 1$ for all $n \in F$.

(ii) There does *not* exist a subset $B \subseteq \mathbb{N}$ such that $|B \cap T_n| = 1$ for all $n \in \mathbb{N}$.