

1. An urn contains seven red balls, numbered 1 through 7, and three blue balls, numbered 1 through 3. Two balls are drawn out, without replacement. Let E be the event that the two balls are the same color, and F the event that their numbers are both even. (a) Find $P(E)$, $P(F)$, $P(EF)$, $P(E \cup F)$, and $P(F|E)$. **Express your answers as simplified fractions.**

Solution: The sample space has $\binom{10}{2}$ points.

(i) We can choose two red balls in $\binom{7}{2}$ ways and two blue balls in $\binom{3}{2}$ ways, so $P(E) =$

$$\left[\binom{7}{2} + \binom{3}{2} \right] / \binom{10}{2} = \frac{8}{15}.$$

(ii) We can choose two even balls in $\binom{4}{2}$ ways, so $P(F) = \binom{4}{2} / \binom{10}{2} = \frac{2}{15}$.

(iii) There are three red balls with even numbers and to get two even balls of the same color we must choose two of these, so $P(EF) = \binom{3}{2} / \binom{10}{2} = \frac{1}{15}$.

(iv) $P(E \cup F) = P(E) + P(F) - P(EF) = \frac{3}{5}$.

(v) $P(F|E) = P(EF)/P(E) = \frac{1}{8}$.

(b) Are E and F independent events? Justify your answer from the definition of independence.

Solution: For E and F to be independent we must have $P(F|E) = P(F)$, or equivalently $P(EF) = P(E)P(F)$. Neither of these holds, so E and F are not independent.

2. A poker hand of five cards is dealt from a standard deck of 52 (without replacement). Find the probability that (a) the hand contains cards of five different ranks; (b) the hand contains four cards of one suit and one of another; (c) the hand contains four cards of one suit and one of another, if it is known that it contains cards of five different ranks.

Solution: Let E be “five different ranks” and F be “four of one suit, one of another.” We will use *unordered* samples, so the sample space has $|S| = \binom{52}{5}$ elements. Many people used this as the denominator in their probability computations but then used *ordered* samples in the numerator; you can’t mix things this way.

(a) Choose the ranks in $\binom{13}{5}$ ways, then one suit for each rank in 4^5 ways, so that

$$P(E) = \binom{13}{5} 4^5 / \binom{52}{5}.$$

(b) Choose the suit in which to have four cards in 4 ways, then four ranks in that suit in $\binom{13}{4}$ ways, then the extra card from the 39 not in that suit: $P(F) = 4 \cdot \binom{13}{4} \cdot 39 / \binom{52}{5}$.

(c) The problem asks for the conditional probability $P(F|E) = P(EF)/P(E)$. We have $P(E)$ from A, so we need $P(EF)$, the probability of having five different ranks, four in one

suit and one in another. This is similar to (b): we choose the suit to have four cards in 4 ways, then choose their ranks in $\binom{13}{4}$ ways, choose a suit and a rank (different from those

already chosen) for the remaining card in $3 \cdot 9$ ways. Thus $P(EF) = 4 \cdot \binom{13}{4} \cdot 3 \cdot 9 / \binom{52}{5}$.

Dividing by $P(E)$ gives $P(F|E) = 4 \cdot \binom{13}{4} \cdot 3 \cdot 9 / \left[\binom{13}{5} \cdot 4^5 \right]$.

3. A coin is flipped repeatedly until it shows heads for the second time.

(a) What is the sample space for this experiment?

Solution: The sample space consists of all sequences of the symbols T (tails) and H (heads) in which exactly two H 's appear, one of which is the last in the sequence:

$$S = \{HH, HTH, THH, TTHH, THTH, HTTH, \dots\} = \{T^i HT^j H \mid i, j \geq 0\},$$

where of course T^i means a string of i T 's.

(b) Suppose that the coin shows heads on any single flip with probability p and that the flips are independent. What is the probability that the coin is flipped a total of n times?

Solution: The probability of any specific sequence in S consisting of exactly n symbols is $(1-p)^{n-2}p^2$, since the flips are independent. There are $n-1$ sequences of length n , since the first H can appear in any of $n-1$ positions in a string of $(n-2)$ T 's. Thus $P(n \text{ tosses to get two heads}) = (n-1)(1-p)^{n-2}p^2$.

4. A drug testing program, working with a group of patients suffering from a certain disease, treats 20% of them with drug A, 30% with drug B, and 50% with drug C. Drug A has a cure rate of 50%, drug B of 60%, and drug C of 90%. If a certain patient recovers, what is the probability he was treated with drug A? You may leave your answer as an unsimplified arithmetic expression.

Solution: Define events A , B , and C that the corresponding drug was used, and E that the patient recovers. then by Bayes' theorem,

$$\begin{aligned} P(A|E) &= \frac{P(EA)}{P(E)} = \frac{P(E|A)P(A)}{P(E|A)P(A) + P(E|B)P(B) + P(E|C)P(C)} \\ &= \frac{(0.5)(0.2)}{(0.5)(0.2) + (0.6)(0.3) + (0.9)(0.5)} = \frac{10}{73}. \end{aligned}$$

5. A certain game uses a coin which, when flipped, shows heads one-third of the time. The coin is flipped twice and the player receives X dollars, where X is three times the number of heads which appear in the two flips.

(a) Find the possible values of X and their probabilities.

Solution: Either 0, 1, or 2 heads may appear. The probabilities of these events are: 0 heads, $(2/3)^2 = 4/9$; 1 head, $2(2/3)(1/3) = 4/9$ (the factor 2 comes because we may get 1 head either as HT or TH); 2 heads, $(1/3)^2 = 1/9$. The player may receive $X = \$0$, with probability $4/9$, $X = \$3$, with probability $4/9$, or $X = \$6$, with probability $1/9$.

(b) Find $E[X]$ and $\text{Var}(X)$; simplify your answers.

Solution: From (a), $E[X] = 0 \cdot (4/9) + 3 \cdot (4/9) + 6 \cdot (1/9) = 2$, $E[X^2] = 0 \cdot (4/9) + 9 \cdot (4/9) + 36 \cdot (1/9) = 8$, and $\text{Var}(X) = E[X^2] - E[X]^2 = 4$.

(c) Let F be the cumulative distribution function of X . Find $F(2)$ and $F(3)$.

Solution: In general, $F(x) = P(X \leq x)$. Here $F(2) = P(X \leq 2) = P(X = 0) = 4/9$ and $F(3) = P(X \leq 3) = P(X = 0) + P(X = 3) = 8/9$.

(d) If the player must pay \$2.50 each time that she plays the game, will she come out ahead or behind in the long run? Explain.

Solution: Behind. She wins an average of $E[X] = \$2.00$ each time she plays, so if she must pay \$2.50 to play she will lose an average of \$0.50 each time.

6. Rose and Bob play a game in which each in turn draws a ball from the urn containing one red and four blue balls, and then replaces it; Rose draws first.

(i) Suppose that Rose wins if she draws a red ball before Bob draws a blue ball, and Bob wins if he draws a blue ball before Rose draws a red ball. What is the probability that Rose wins? Hint: condition on the results of the first two draws.

Solution: Let E be the event that Rose wins. On the first two draws one of three things may happen: Rose may draw a red ball (R_1 ; then Rose wins and the game is over), Rose may draw a blue and Bob a red (B_1R_2 , then they are in effect starting over), or Rose may draw a blue ball and Bob also a blue (B_1B_2 , so Bob wins). Conditioning gives

$$\begin{aligned} P(E) &= P(E|R_1)P(R_1) + P(E|B_1R_2)P(B_1R_2) + P(E|B_1B_2)P(B_1B_2) \\ &= 1 \cdot \frac{1}{5} + P(E) \cdot \frac{4}{25} + 0 \cdot \frac{16}{25} = \frac{1}{5} + P(E) \cdot \frac{4}{25}. \end{aligned}$$

Solving for $P(E)$ gives $P(E) = 5/21$.

(b) If the rules are modified so that Rose wins if she draws a red ball before Bob draws a blue ball *twice*, what then is the probability that Rose wins?

Solution: Let F be the event that Rose wins this new game. The analysis is almost the same as in (a), but now after B_1B_2 Bob does not win; rather, it is as if they start playing the game in (a); thus $P(F|B_1B_2) = P(E) = 5/21$. So

$$\begin{aligned} P(F) &= P(F|R_1)P(R_1) + P(F|B_1R_2)P(B_1R_2) + P(F|B_1B_2)P(B_1B_2) \\ &= 1 \cdot \frac{1}{5} + P(F) \cdot \frac{4}{25} + P(E) \cdot \frac{16}{25} = \frac{1}{5} + P(F) \cdot \frac{4}{25} + \frac{16}{105}. \end{aligned}$$

Solving for $P(F)$ yields $P(F) = 185/441$.