Calculators will not be used; either the arithmetic will be quite simple or you will be asked to leave answers in unsimplified form. You will be given a copy of Table 5.1 of Ross so that you can answer questions about the normal distribution, and the following formulas:

Binomial: $P\{X = k\} = \binom{n}{k} p^k(1-p)^{n-k}$, $k = 0, 1, \ldots, n$. $E[X] = np$, $\mathrm{Var}(X) = np(1-p)$.

Geometric: $P\{X = k\} = p(1-p)^{k-1}$, $k = 1, 2, \ldots$. $E[X] = 1/p$, $\mathrm{Var}(X) = (1-p)/p^2$.

Poisson: $P\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda}$, $k = 0, 1, 2, \ldots$. $E[X] = \lambda$, $\mathrm{Var}(X) = \lambda$.

Exponential: $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$. $E[X] = 1/\lambda$, $\mathrm{Var}(X) = 1/\lambda^2$.

Normal: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$. $E[X] = \mu$, $\mathrm{Var}(X) = \sigma^2$.

1. A certain continuous random variable X has density

$$f_X(x) = \begin{cases} cx, & \text{if } 0 \leq x \leq 1, \\ 0, & \text{if } x < 0 \text{ or } x > 1, \end{cases}$$

for some constant c.

(a) Find c. (b) Find $\mathrm{P}\{X \geq \frac{1}{3}\}$ and $\mathrm{P}\{X = \frac{1}{3}\}$. (c) Find $E[X]$ and $\mathrm{Var}(X)$.

Suppose now that Y is a second continuous random variable which is uniformly distributed on the interval $[0, 1]$, and that X and Y are independent.

(d) Write down carefully the joint density $f(x, y)$. Be sure both to specify where $f(x, y)$ is zero and to give a formula for it where it is non-zero.

(e) Find $\mathrm{P}\{X \geq Y\}$.

2. A binomial random variable X with parameters n and p is to be approximated by a Poisson random variable Y with parameter np. Under what restrictions on n and p is this approximation reasonable? Show that, when these restrictions hold, Y has the "correct" mean and almost the "correct" variance.

3. Robert plays a game in which his chance of winning is 1/5. Using a normal approximation with half integer correction, estimate the probability if he plays the game 100 times he will win exactly 25 times.

4. Alice, Bruce, and Cynthia are playing darts, using the disk $x^2 + y^2 \leq 4$ as a target. They always manage to hit the target, but the x and y components of their impact points have different joint distributions, denoted f_A, f_B, and f_C, respectively; for $x^2 + y^2 \leq 4$,

$$f_A(x, y) = c_A(4 - x^2 - y^2), \qquad f_B(x, y) = c_B, \qquad f_C(x, y) = c_C(x^2 + y^2).$$

(a) Assuming that all are aiming for the center of the target, which one is the best shot? The worst shot?

(b) Find c_A, c_B, and c_C.

(c) The game is scored by giving 4 points for a hit inside the circle $x^2 + y^2 = 1$, and 1 point for a hit outside that circle. What is the expected value of the number of points scored by each player on a throw?

(Hint: Part (a) can be done by inspection. For Alice and Cynthia, parts (b) and (c) require a little work (use polar coordinates), but for Bruce, they can be done almost by inspection.)

5. A certain component has lifetime T, where T is a positive random variable, measured in days, with density $f(t) = Kte^{-2t}$.

(a) Find K.

(b) Suppose it is known that the component has lasted one day. What is the probability that it will last two more days?

6. Let X be a random variable which is uniformly distributed on the interval $[-1,3]$. Compute the density for $Y = X^4$. Hint: you should find different formulas for the density in the two regions $0 \le y \le 1$ and $1 \le y \le 81$.

7. Let X be a binomial random variable with $n = 3$ and $p = .5$, and let Y be a geometric random variable with parameter $p = .5$. Suppose that X and Y are independent.

(a) Find the joint probability mass function $p(x, y)$ of X and Y. Write out the values in tabular form for, say, $y = 1, 2, 3,$ and 4 and all values of X.

(b) Find $P\{X < Y\}$.

(c) Find the probability mass function for $X + Y$.

8. Cars pass a certain point on a lonely road according to a Poisson process, with an average rate of 3 cars per hour. Find the probability that an observer will see exactly three cars pass in one hour of observation

(a) If three cars pass in the first half hour;

(b) If no cars pass in the first half hour.

9. By definition, a "hundred year flood" on a river is a flood which is so severe that it happens, on the average, once every hundred years. Find the probability that there will be exactly three hundred year floods on the Raritan river between 2051 and 2250 (inclusive), both exactly and by using a suitable Poisson approximation. Assume that at most one such flood can occur in any year and that floods in different years are independent.

10. Let X and Y be independent random variables, exponentially distributed with parameters λ and μ, respectively.

(a) Find $P\{X > 2Y\}$.

(b) Find the probability density for the random variable $Z = X + Y$.

11. The joint mass function of two discrete random variables X and Y is given by the following table:

$x \backslash y$	0	1	2
0	0.1	0.1	0.3
1	0	0.2	0
2	0.1	0.2	0

(a) Find the (marginal) distributions—that is, the mass functions—of X and Y.

(b) Find the probability mass function for the random variable $Z = X - Y$.

(c) Find the expectation and variance of X and of Z.

12. (a) Two random variables X and Y are jointly uniformly distributed on the disk $X^2 + Y^2 \le 1$. Find there marginal disturbutions. Are they independent?

(b) Is it possible to find a joint distribution for two random variables X and Y such that they are independent and always $X^2 + Y^2 \le 1$?

13. Two friends agree to meet every Monday evening on the corner of College Avenue and Seminary Place. Each arrives at a random time between 6:00 PM and 6:30 PM. Dorothy always waits exactly five minutes for Evan to show up, then leaves if he is not there; Evan, more patient, will wait 10 minutes for Dorothy. On what fraction of the Mondays do they actually meet?

Some answers (not complete, not checked carefully; be suspicious):

1. (a) 2; (b) 8/9, 0; (c) 2/3, 1/18; (e) 2/3.
3. 0.0457.
4. (b) $1/(8\pi)$, $1/(4\pi)$, $1/(8\pi)$; (c) 37/16, 7/4, 19/16.
5. (a) 4; (b) $7e^{-4}/3$.
6. $y^{-3/4}/8$, $0 \le y \le 1$, $y^{-3/4}/16$, $1 < y \le 81$.
7. (b) 27/64; (c) $p_{X+Y}(k)$: 1/16, 7/32, 19/64 for $k = 1, 2, 3$; $27/2^{k+3}$ for $k \ge 4$.
8. (a) $e^{-3/2}$; (b) $9e^{-3/2}/16$.
9. Poisson: $4e^{-2}/3$; Exact: $\binom{200}{3}99^{197}/100^{200}$.
10. (a) $\mu/(\mu + 2\lambda)$; (b) $\mu\lambda(e^{-\mu z} - e^{-\lambda z})/(\lambda - \mu)$
11. $E[X] = 0.8$, $\mathrm{Var}(X) = 0.76$, $E[Z] = -0.3$, $\mathrm{Var}(Z) = 1.81$.
12. (a) $2\sqrt{1 - x^2}/\pi$, no; (b) yes.
13. 31/72