

1. An urn contains $r = r_1 + r_2$ packages of candy; of these, r_1 contain one piece and r_2 contain three pieces.
 (a) If a package were to be drawn from the urn at random, what is the probability it would contain one piece of candy? Three pieces? If two packages were drawn (without replacement), what are the probabilities of obtaining altogether two, four, or six pieces of candy?

For (b) and (c) below, we suppose that a sample of n packages ($n \leq r$) is drawn from the urn, one at a time, without replacement.

(b) Let X_i be the number of pieces of candy in the i^{th} package drawn. Find $E[X_i]$, $\text{Var}(X_i)$, and $\text{Cov}(X_i, X_j)$ for $i \neq j$. Hint: use your results from (a).

(c) Let X be the total number of pieces of candy obtained. Find $E[X]$ and $\text{Var}(X)$. Check your answer by inspection when $n = r$.

2. Let X and Y be continuous random variables with joint density

$$f(x, y) = \begin{cases} cx^2ye^{-xy}, & \text{if } 1 \leq x \leq 2 \text{ and } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

(a) If we know that X takes value $3/2$, what is the (conditional) distribution of Y ? No calculations are needed: this is just a question of recognition.

Hint for (b)–(c): Set up as double integrals and do the y integral first.

(b) Find the constant c .

(c) Find $E[X]$, $E[Y]$, and $\text{Cov}(X, Y)$.

3. (a) Xerxes and Yvonne play a game in which each, independently, chooses a real number between 0 and 2; their choices are called X and Y , respectively. Xerxes wins if his choice is at least twice Yvonne's. Suppose that both choose "at random", so that their choices are uniformly distributed on the interval $[0, 2]$. What is the probability that Xerxes wins?

(b) Suppose that each time the game is played Yvonne contributes \$2.00 and Xerxes \$1.00 to a pot, with the winner of the game collecting the entire \$3.00, and that they play the game 100 times in this fashion. Use the central limit theorem to estimate the probability that Yvonne comes out at least \$35.00 ahead. (The half-integer correction would be a bit subtle here; don't worry about using it.)

4. Alice and Bob play a game with a coin which shows heads with probability p . Alice flips the coin three times, and then Bob does also. The one with the most heads (out of his or her three flips) wins.

(a) What is the probability that the two players have a total of at least five heads?

(b) What is the probability that a tie occurs?

(c) What is the probability that Bob has already won the game after his first flip?

5. Three urns are numbered 1 through 3; urn k contains k balls numbered 1 through k . We select an urn at random, draw a ball from it, note the number of the ball, replace the ball, and then draw again from the same urn. If it is known that the first ball drawn has number 1, find (a) the probability mass function of the number of the selected urn; (b) the expected value of the number of the second ball drawn.

6. Let X and Y be independent continuous random variables with densities

$$f_X(x) = \begin{cases} e^{-x}, & \text{if } x \geq 0, \\ 0, & \text{otherwise;} \end{cases} \quad f_Y(y) = \begin{cases} 3e^{-3y}, & \text{if } y \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

Find the density function $f_Z(z)$ for $Z = X + Y$ (a) by computing $F_Z(z)$; (b) using (3.2) of Chapter 6.

7. Let X_1, X_2, \dots, X_{25} be independent random variables, each of which is uniformly distributed on the interval $[0, 2]$, and let $X = \sum_{i=1}^{25} X_i$.

(a) Find the mean and variance of X .

(b) Use the central limit theorem to estimate the probability that $|X - 24| \leq 3$.

(c) Use Chebyshev's inequality to find a number a such that you are absolutely sure that $P\{|X - E[X]| \geq a\} \leq 0.1$.

8. A single die is rolled; let us denote the number shown by X . Then a (possibly) biased coin, with probability $X/6$ of showing heads, is flipped; the number of heads (0 or 1) obtained is denoted Y . (For example, if a 3 is rolled then a fair coin is flipped.)

- (a) Find the joint probability mass function, and the marginal probability mass functions, of X and Y .
- (b) Find $E[X]$, $E[Y]$, $\text{Var}(X)$, $\text{Var}(Y)$, and $\text{Cov}(X, Y)$.
- (c) Find the probability mass function of the random variable $Z = X + Y$.

9. A modified form of poker is played with an ordinary deck of 52 playing cards, but hands contain six cards. Calculate the probability that a randomly dealt hand contains: (i) four cards of the same rank and two from different ranks; (ii) three pairs, each of a different rank; (iii) two triples, each containing three cards of the same rank; (iv) only two ranks; (v) six cards in sequence; (vi) six cards, all of different ranks; (vii) a triple, a pair, and a single card, all of different ranks.

10. (a) A hat contains n cards numbered 1 to n ; one card is drawn at random and its number is denoted X . Calculate the moment generating function $M_X(t)$ of X . (Hint: review the formula for the sum of a finite geometric series.)

(b) If the experiment is repeated k times (with replacement) and Y is the sum of the numbers of all cards drawn, find $M_Y(t)$.

11. (a) State Markov's inequality as a theorem, defining all symbols used and including all necessary hypotheses.

(b) State Chebyshev's inequality as a theorem, defining all symbols used and including all necessary hypotheses.

12. "I've always believed that 10% of the financiers on Wall Street are shady," grumbled the S.E.C. inspector, "but the difficulty is to know which ones they are. Take the Amalgamated Widgets crash. An honest investor would have had only a 20% chance of coming out unscathed, but there is a 70% chance that a crook would have had inside information and gotten out in time. Now Smith didn't lose any money" Assuming that the inspector's estimates are correct, what is the probability that Smith is honest?

13. A large urn contains N balls of each of 20 different colors (that is, a total of $20N$ balls). 10 balls are selected at random; we let X be the total number of different colors obtained, and write $X = \sum_{i=1}^{20} X_i$, with X_i a Bernoulli random variable indicating whether or not the i^{th} color is obtained.

(a) Suppose that the selection is without replacement. Find $E[X]$ and $\text{Var}(X)$. (Hint: compute $E[X_i]$ by computing $P\{X_i = 0\}$ using equally likely outcomes; do $E[X_i X_j]$ similarly.)

(b) Find by elementary reasoning the correct values in (a) when $N = 1$, and verify that the answers you obtained there are correct in that case.

(c) Suppose that the selection is with replacement. Find $E[X]$ and $\text{Var}(X)$. Your answer should not depend on N .

(d) Show that your answers in (c) are the $N \rightarrow \infty$ limit of your answers in (a). Explain why this should be true.

14. Each morning, Fred makes a random choice of one of three routes to take to work. After n trips, ($n > 0$), what is the probability that he has traveled each route at least once? Hint: use inclusion-exclusion to calculate the probability of the complementary event.

15. Discuss *one* of the following classical problems in probability, describing the problem (or model, or situation) and giving some details about the solution: gambler's ruin, the problem of the points, coupon collecting, the matching problem, Burron's needle.