

# 1. THE PROBABILITY OF WINNING AT CRAPS

The game is played by rolling a pair of fair dice and adding the numbers on their face. The rules are:

- (1) If the first roll sums to 7 or 11, the game is over and you win. If the first roll sums to 2, 3, or 12, the game is over and you lose.
- (2) If the first roll sums to 4, 5, 6, 8, 9, or 10, that sum becomes your point. After that, the dice are rolled until either a 7 or your point appears, at which time the game is over. If your point appears, you win; if 7 appears you lose.

In order to determine the probability of winning, we must first set up an appropriate sample space. We first compute the probabilities of each of the possible sums. Since the dice are assumed fair, all of the 36 possible pairs of faces are considered equally likely. We must use ordered pairs rather than unordered pairs because, for example, the unordered pair  $\{1, 1\}$  does not occur as frequently as the unordered pair  $\{1, 2\}$

Sum	Pairs giving sum	Probability
2	(1, 1)	$\frac{1}{36}$
3	(1, 2) (2, 1)	$\frac{1}{18}$
4	(1, 3) (2, 2) (3, 1)	$\frac{1}{12}$
5	(1, 4) (2, 3) (3, 2) (4, 1)	$\frac{1}{9}$
6	(1, 5) (2, 4) (3, 3) (4, 2) (5, 1)	$\frac{5}{36}$
7	(1, 6) (2, 5) (3, 4) (4, 3) (5, 2) (6, 1)	$\frac{1}{6}$

Sum	Pairs giving sum	Probability
12	(6, 6)	$\frac{1}{36}$
11	(6, 5) (5, 6)	$\frac{1}{18}$
10	(6, 4) (5, 5) (4, 6)	$\frac{1}{12}$
9	(6, 3) (5, 4) (4, 5) (3, 6)	$\frac{1}{9}$
8	(6, 2) (5, 3) (4, 4) (3, 5) (2, 6)	$\frac{5}{36}$

Let us now compute the probability of the event  $E_n$  that you win in precisely  $n$  rolls. If the game is not over the first round, that is, if  $n \geq 2$ , we let  $E_{n,i}$  be the event that the point is  $i$  and the game is over in precisely  $n$  rolls. Then  $E_n$  is the disjoint union

$$E_n = \bigcup_{i \in \{4,5,6,8,9,10\}} E_{n,i}$$

If  $n = 1$ ,  $E_1$  is the event that a 7 or 11 is rolled the first round.

$$P(E_1) = \frac{1}{6} + \frac{1}{18} = \frac{2}{9}.$$

If  $n = 2$ ,  $E_2$  is the event that the first roll gives a sum in  $\{4, 5, 6, 8, 9, 10\}$  which has probability  $\left(\frac{1}{12} + \frac{1}{9} + \frac{5}{36}\right) \times 2 = \frac{2}{3}$ , and the second roll gives the same sum as the first. This second possibility

depends on the point, so we compute

$$P(E_{2,4}) = P(E_{2,10}) = \frac{1}{12} \times \frac{1}{12}$$

$$P(E_{2,5}) = P(E_{2,9}) = \frac{1}{9} \times \frac{1}{9} \quad .$$

$$P(E_{2,6}) = P(E_{2,8}) = \frac{5}{36} \times \frac{5}{36}$$

If  $n \geq 3$ ,  $E_n$  is the event that the first roll gives a sum in  $\{4, 5, 6, 8, 9, 10\}$ , the second roll gives a sum in  $\{i : 2 \leq i \leq 12, i \neq 7, i \neq \text{the point}\}$ . Here we compute, subtracting the probability that  $i = 7$  and the probability that  $i = \text{the point}$ ,

$$P(E_{n,4}) = P(E_{n,10}) = \frac{1}{12} \times \left(\frac{5}{6} - \frac{1}{12}\right)^{n-2} \times \frac{1}{12} = \left(\frac{1}{12}\right)^2 \times \left(\frac{3}{4}\right)^{n-2}$$

$$P(E_{n,5}) = P(E_{n,9}) = \frac{1}{9} \times \left(\frac{5}{6} - \frac{1}{9}\right)^{n-2} \times \frac{1}{9} = \left(\frac{1}{9}\right)^2 \times \left(\frac{13}{18}\right)^{n-2} \quad .$$

$$P(E_{n,6}) = P(E_{n,8}) = \frac{5}{36} \times \left(\frac{5}{6} - \frac{5}{36}\right)^{n-2} \times \frac{5}{36} = \left(\frac{5}{36}\right)^2 \times \left(\frac{25}{36}\right)^{n-2}$$

Since we know the sum of a geometric series, we can now compute the probability of a win. We use the fact that the probabilities of sums come in pairs in our table at the beginning.

$$\begin{aligned} \sum_{n=1}^{\infty} P(E_n) &= P(E_1) + \sum_{\substack{n=1 \\ i \in \{4, 5, 6, 8, 9, 10\}}}^{\infty} P(E_{n,i}) \\ &= \frac{2}{9} + 2 \cdot \sum_{n=2}^{\infty} \left( \left(\frac{1}{12}\right)^2 \left(\frac{3}{4}\right)^{n-2} + \left(\frac{1}{9}\right)^2 \left(\frac{13}{18}\right)^{n-2} + \left(\frac{5}{36}\right)^2 \left(\frac{25}{36}\right)^{n-2} \right) \\ &= \frac{2}{9} + 2 \cdot \left( \left(\frac{1}{12}\right)^2 \sum_{k=0}^{\infty} \left(\frac{3}{4}\right)^k + \left(\frac{1}{9}\right)^2 \sum_{k=0}^{\infty} \left(\frac{13}{18}\right)^k + \left(\frac{5}{36}\right)^2 \sum_{k=0}^{\infty} \left(\frac{25}{36}\right)^k \right) \\ &= \frac{2}{9} + 2 \cdot \left( \left(\frac{1}{12}\right)^2 \left(\frac{1}{1-3/4}\right) + \left(\frac{1}{9}\right)^2 \left(\frac{1}{1-13/18}\right) + \left(\frac{5}{36}\right)^2 \left(\frac{1}{1-25/36}\right) \right) \\ &= \frac{2}{9} + 2 \cdot \left( \frac{1}{36} + \frac{2}{45} + \frac{25}{396} \right) \\ &= 0.2222222 + 2 \cdot (2.777778 \times 10^{-2} + 4.444444 \times 10^{-2} + 6.313131 \times 10^{-2}) \\ &= 0.4929293 \end{aligned}$$

The game of craps often involves side bets. After the first roll, one knows the point (or that the game is over), and the probability of a win changes because the point is known. Let us look at a little table of the probabilities of a win looked at after the first roll has been taken.

The event  $E_1$  is the intersection of two events, namely  $E$  = the event that you win and  $F$  = the event that the game ends in 1 roll. For  $n \geq 2$ , the event  $E_{n,i}$  is the intersection of two events, namely  $E$  and the event that the point is  $i$  and the game ends at roll  $n$ . The rows and columns are labeled by events so that the entry in row  $\alpha$  and column  $\beta$  is the probability of the event labeling the row  $\alpha$

intersect the event labeling column  $\beta$ .

	Win	Lose
End in 1 roll	$\frac{2}{9}$	$\frac{1}{9}$
End in $n \geq 2$ rolls, pt = 4	$\left(\frac{1}{12}\right)^2 \left(\frac{3}{4}\right)^{n-2}$	$\left(\frac{1}{12}\right) \left(\frac{3}{4}\right)^{n-2} \left(\frac{1}{6}\right)$
End in $n \geq 2$ rolls, pt = 5	$\left(\frac{1}{9}\right)^2 \left(\frac{13}{18}\right)^{n-2}$	$\left(\frac{1}{9}\right) \left(\frac{13}{18}\right)^{n-2} \left(\frac{1}{6}\right)$
End in $n \geq 2$ rolls, pt = 6	$\left(\frac{5}{36}\right)^2 \left(\frac{25}{36}\right)^{n-2}$	$\left(\frac{5}{36}\right) \left(\frac{25}{36}\right)^{n-2} \left(\frac{1}{6}\right)$
End in $n \geq 2$ rolls, pt = 8	$\left(\frac{5}{36}\right)^2 \left(\frac{25}{36}\right)^{n-2}$	$\left(\frac{5}{36}\right) \left(\frac{25}{36}\right)^{n-2} \left(\frac{1}{6}\right)$
End in $n \geq 2$ rolls, pt = 9	$\left(\frac{1}{9}\right)^2 \left(\frac{13}{18}\right)^{n-2}$	$\left(\frac{1}{9}\right) \left(\frac{13}{18}\right)^{n-2} \left(\frac{1}{6}\right)$
End in $n \geq 2$ rolls, pt = 10	$\left(\frac{1}{12}\right)^2 \left(\frac{3}{4}\right)^{n-2}$	$\left(\frac{1}{12}\right) \left(\frac{3}{4}\right)^{n-2} \left(\frac{1}{6}\right)$

We can simplify this by countable additivity by adding  $\sum_{n=0}^{\infty} E_{n, i}$  which is a geometric series to get the table of interest, and then add the row sums and the column sums as indicated.

	Win	Lose	Prob of row event
Over in single roll	$\frac{2}{9}$	$\frac{1}{9}$	$\frac{1}{3}$
Point = 4	$\left(\frac{1}{12}\right)^2 \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^{n-2} = \frac{1}{36}$	$\sum_{n=0}^{\infty} \left(\frac{1}{12}\right) \left(\frac{3}{4}\right)^{n-2} \left(\frac{1}{6}\right) = \frac{1}{18}$	$\frac{1}{12}$
Point = 5	$\left(\frac{1}{9}\right)^2 \sum_{n=0}^{\infty} \left(\frac{13}{18}\right)^{n-2} = \frac{2}{45}$	$\sum_{n=0}^{\infty} \left(\frac{1}{9}\right) \left(\frac{13}{18}\right)^{n-2} \left(\frac{1}{6}\right) = \frac{2}{30}$	$\frac{1}{9}$
Point = 6	$\left(\frac{5}{36}\right)^2 \sum_{n=0}^{\infty} \left(\frac{25}{36}\right)^{n-2} = \frac{25}{396}$	$\sum_{n=0}^{\infty} \left(\frac{5}{36}\right) \left(\frac{25}{36}\right)^{n-2} \left(\frac{1}{6}\right) = \frac{5}{66}$	$\frac{5}{36}$
Point = 8	$\sum_{n=0}^{\infty} \left(\frac{5}{36}\right)^2 \left(\frac{25}{36}\right)^{n-2} = \frac{25}{396}$	$\sum_{n=0}^{\infty} \left(\frac{5}{36}\right) \left(\frac{25}{36}\right)^{n-2} \left(\frac{1}{6}\right) = \frac{5}{66}$	$\frac{5}{36}$
Point = 9	$\sum_{n=0}^{\infty} \left(\frac{1}{9}\right)^2 \left(\frac{13}{18}\right)^{n-2} = \frac{2}{45}$	$\sum_{n=0}^{\infty} \left(\frac{1}{9}\right) \left(\frac{13}{18}\right)^{n-2} \left(\frac{1}{6}\right) = \frac{2}{30}$	$\frac{1}{9}$
Point = 10	$\sum_{n=0}^{\infty} \left(\frac{1}{12}\right)^2 \left(\frac{3}{4}\right)^{n-2} = \frac{1}{36}$	$\sum_{n=0}^{\infty} \left(\frac{1}{12}\right) \left(\frac{3}{4}\right)^{n-2} \left(\frac{1}{6}\right) = \frac{1}{18}$	$\frac{1}{12}$
Prob of column event	$\frac{244}{495}$	$\frac{251}{495}$	1

A glance at this table tells us something very interesting. Although your chances of winning are very close to 0.5 before the game begins, if the game has not ended at the end of the first round, they vary by what the point is. Let us say that your point is 4. You now know that, so effectively the probability that the point is 4 is 1. So you should divide every entry in the row labeled “Point = 4” by  $P(\text{point} = 4)$ . The resulting entries are the probability of a win or a loss given that the point is 4, denoted  $P(\text{col event} \mid \text{point} = 4)$ . In general, the probability of event  $F$  given event  $G$ ,  $P(F \mid G) = \frac{P(FG)}{P(G)}$ . If you are given that  $G$  is true, then you want  $P(G \mid G)$  to be 1.

In our example,  $P(\text{win} \mid \text{point} = 4) = \frac{1/36}{1/12} = \frac{1}{3}$ ;  $P(\text{win} \mid \text{point} = 5) = \frac{2/45}{1/9} = \frac{2}{5}$ ;  $P(\text{win} \mid \text{point} = 6) = \frac{25/396}{5/36} = \frac{5}{11}$ . These are the probabilities by which side bets after the first roll are determined, not the original 0.4929293.