

1. P294:45. The probability density function is 1, so the requested probability would be the volume of the solid satisfying the inequalities  $0 \leq X_i \leq 1$  for  $i \in \{1, 2, 3\}$ , and  $X_i + X_j \leq X_k$  where  $X_k = \max\{X_i, X_j, X_k\}$  and  $i, j, k$  are all distinct. We first look at the region where  $X_1 < X_2 < X_3$ . Then the corresponding volume is  $\int_0^{1/2} \int_{x_1}^{1-x_1} \int_{x_1+x_2}^1 dx_3 dx_2 dx_1 = \frac{1}{12}$ . There are 6 disjoint volumes of this size corresponding to the 6 ways of writing the random variables  $X_1, X_2, X_3$  in ascending order, so the total probability is  $6 \times \frac{1}{12} = \frac{1}{2}$ .
2. P295:49. If  $\{X_i : 1 \leq i \leq 5\}$  are independent and identically distributed exponential random variables with the parameter  $\lambda$ , compute:

$$(a) P\{\min\{X_i\} \geq a\} = P\{X_i \geq a \text{ for all } i\} = \begin{cases} \prod_{i=1}^5 \int_a^\infty \lambda e^{-\lambda t} dt = (e^{-a\lambda})^5 = e^{-5\lambda a} & a > 0 \\ 0 & a \leq 0 \end{cases} \text{ so}$$

$$P\{\min\{X_i\} \leq a\} = 1 - e^{-5\lambda a}.$$

Thus  $\min\{X_i\}$  is an exponential r.v. with parameter  $5\lambda$ .

$$(b) P\{\max\{X_i\} \leq a\} = P\{X_i \leq a \text{ for all } i\} = \begin{cases} \prod_{i=1}^5 \int_0^a \lambda e^{-\lambda t} dt = (-e^{-\lambda}|_0^a)^5 = (1 - e^{-a\lambda})^5 & a > 0 \\ 0 & a \leq 0 \end{cases}$$

3. p296:8. Let  $X$  and  $Y$  be independent continuous random variables with respective hazard rate functions  $\lambda_X(t)$  and  $\lambda_Y(t)$ . Set  $W = \min(X, Y)$ . The hazard rate function of the random variable  $U$  is defined as

$$\lambda_U(t) = \frac{f_U(t)}{1 - F_U(t)}$$

(see p215) where  $f_U$  is the probability density function and  $F_U$  is the cumulative distribution function of  $U$ .

- (a) Determine the distribution function of  $W$  in terms of those of  $X$  and  $Y$ .

$1 - F_W(t) = P\{\min\{X, Y\} > t\} = P\{X > t \text{ and } Y > t\} = (1 - F_X(t))(1 - F_Y(t))$  by independence, so

$$F_W(t) = 1 - (1 - F_X(t))(1 - F_Y(t)) = F_X(t) + F_Y(t) - F_X(t)F_Y(t)$$

and  $f_W(t) = f_X(t) + f_Y(t) - f_X(t)F_Y(t) - F_X(t)f_Y(t)$  by the product rule since  $\frac{dF_U(t)}{dt} = f_U(t)$ .

- (b) Show that  $\lambda_W(t)$ , the hazard rate function of  $W$ , is given by

$$\lambda_W(t) = \lambda_X(t) + \lambda_Y(t)$$

By (a),

$$\begin{aligned} \lambda_W(t) &= \frac{f_X(t) + f_Y(t) - f_X(t)F_Y(t) - F_X(t)f_Y(t)}{(1 - F_X(t))(1 - F_Y(t))} \\ &= \frac{f_X(t) - f_X(t)F_Y(t)}{(1 - F_X(t))(1 - F_Y(t))} + \frac{f_Y(t) - F_X(t)f_Y(t)}{(1 - F_X(t))(1 - F_Y(t))} \\ &= \frac{f_X(t)}{(1 - F_X(t))} + \frac{f_Y(t)}{(1 - F_Y(t))} = \lambda_X(t) + \lambda_Y(t) \end{aligned}$$

4. p298:10. The lifetimes of batteries are independent exponential random variables, each having parameter  $\lambda$ . A flashlight needs two batteries to work. If one has a flashlight and a stockpile of  $n$  batteries, what is the distribution of time that the flashlight can operate?

This problem appears to be somewhat unclear exactly as stated in the book, so I have emailed a clarification. Here I will indicate first why it is a good problem if one interprets it as it should be interpreted in practice rather than taking it literally. I will then show why taking it literally leads to an insolvable problem.

We can assume that the batteries have a sharp drop from full voltage to no voltage, and that they do not drain when they are in storage. Problem: nothing whatsoever is said in the problem about the possibility

of determining which battery is at fault when the flashlight stops operating. So I will assume that when the flashlight fails, **both** batteries are changed (a very good idea in practice). In this case, if  $n = 2k$  or  $n = 2k + 1$ , there will be  $k$  changes of batteries. Let  $X_i$  be the r.v. giving the operating lifetime of the  $i^{\text{th}}$  pair of batteries. Then  $X_i$  is the minimum of the lifetimes of the  $i^{\text{th}}$  and  $i + 1^{\text{st}}$  batteries. The minimum of two independent exponential r.v.'s  $U$  and  $V$  with parameters  $\lambda$  and  $\lambda'$  is an exponential r.v. with parameter  $\lambda + \lambda'$  since  $P\{\min\{U, V\} > t\} = P\{U > t \text{ and } V > t\} = \left(\int_t^\infty \lambda e^{-\lambda u} du\right) \left(\int_t^\infty \lambda e^{-\lambda' u} du\right) = e^{-(\lambda+\lambda')t}$  by independence, and

$$F_{\min\{U, V\}}(t) = 1 - e^{-(\lambda+\lambda')t}$$

$$f_{\min\{U, V\}}(t) = (\lambda + \lambda') e^{-(\lambda+\lambda')t}$$

so  $X_i$  is an exponential r.v. with parameter  $2\lambda$ . Thus  $E[X_i] = \frac{1}{2\lambda}$  and

$$E\left[\sum_{i=1}^k X_i\right] = \sum_{i=1}^k E[X_i] = \left\lfloor \frac{n}{2} \right\rfloor \times \frac{1}{2\lambda}$$

where  $\left\lfloor \frac{n}{2} \right\rfloor$  denotes the greatest integer  $\leq \frac{n}{2}$ .

Now for the problem if you can find and change only one dead battery if the other has life left in it. First of all, it is not clear that the lifetime of the two batteries in the flashlight are independent. But let us assume that they are.

If  $n = 3$ , then for  $Y$  the time the flashlight operates,

$$Y = \min(\max\{X_1, X_2\}, \min\{X_1, X_2\} + X_3)$$

since you put in 2 batteries and replace the one that dies first with the remaining battery. As above,  $\min\{X_1, X_2\}$  is an exponential r.v. with parameter  $2\lambda$ . Now look at the sum  $W = \min\{X_1, X_2\} + X_3$ . As in the proof of Proposition 3.1 on page 262, for  $a > 0$ ,

$$F_W(a) = \frac{1}{\Gamma(1)\Gamma(1)} \int_0^a \lambda e^{-\lambda(a-t)} 2\lambda e^{-2\lambda t} dt$$

$$= 2\lambda^2 \int_0^a e^{-\lambda(a+t)} dt = 2\lambda^2 e^{-\lambda a} \int_0^a e^{-\lambda t} dt$$

$$= 2\lambda^2 e^{-\lambda a} \frac{1}{\lambda} (e^{-\lambda t}) \Big|_0^a = 2\lambda e^{-\lambda a} (1 - e^{-\lambda a})$$

and this is not the cumulative distribution function of a named random variable. We can also compute  $F_{\max\{X_1, X_2\}}(t) = P\{X_1 \leq t, X_2 \leq t\} = \left(\int_0^t \lambda e^{-\lambda u} du\right)^2 = e^{-2t\lambda} (e^{t\lambda} - 1)^2$  and  $f_{\max\{X_1, X_2\}}(t) = \frac{de^{-2t\lambda} (e^{t\lambda} - 1)^2}{dt} = e^{-2t\lambda} (2\lambda e^{t\lambda} - 2\lambda)$ . This is also not a named distribution.

Now for  $Y = \min\{\max\{X_1, X_2\}, W\}$ ,  $1 - F_Y(y) = P\{Y \geq y\} = P\{\max\{X_1, X_2\} \geq y, W \geq y\}$ , and if  $\max\{X_1, X_2\}$  and  $W$  are independent, as they appear to be since information about the larger of  $X_1$  and  $X_2$  tells you nothing about  $X_3$  and hence nothing about  $W$ , then this equals  $(1 - F_{\max\{X_1, X_2\}}(y)) (1 - F_W(y))$  where we know both of the indicated cumulative distribution functions, so we can come up with a very messy cumulative distribution function.

I will not attempt a similar analysis for  $n = 4$ . For example, you may have one of the original 2 batteries used outliving the sum of the lifetimes of the other 3, or you may have to change each of the original two batteries one time. So you no longer can express the time of operation of the flashlight as a single minimum. It will be a sum of minima, and they will not be independent.

5. p298:14. Suppose  $X$  and  $Y$  are independent geometric r.v.'s with the same parameter  $p$ . Find  $P\{X = i \mid X + Y = n\}$ .  $P\{X + Y = n\} = \sum_{i=1}^{n-1} P\{X = i \text{ and } Y = n - i\} = \sum_{i=1}^{n-1} q^{i-1} p q^{n-i-1} p = (n-1) q^{n-2} p^2$  since  $X$  and  $Y$  are independent. Now for a fixed  $i$ ,  $P\{X = i \text{ and } X + Y = n\} = q^{i-1} p q^{n-i-1} p = q^{n-2} p^2$  so

$$P\{X = i \mid X + Y = n\} = \frac{q^{n-2} p^2}{(n-1) q^{n-2} p^2} = \frac{1}{n-1} \quad \text{for } n \geq 2, 1 \leq i \leq n-1$$

6. P380:3. First a little harder than necessary way of computing this which shows you are integrating over the entire unit square without drawing a picture (not a good idea, but pictures are hard to draw in a .pdf file).

$$\begin{aligned}
 E[|X - Y|^\alpha] &= \int_0^1 \int_0^y (y - x)^\alpha dx dy + \int_0^1 \int_y^1 (x - y)^\alpha dx dy \\
 &= \int_0^1 \left. \frac{-(y - x)^{\alpha+1}}{\alpha + 1} \right|_0^y dy + \int_0^1 \left. \frac{(x - y)^{\alpha+1}}{\alpha + 1} \right|_y^1 dx \\
 &= \int_0^1 \frac{y^{\alpha+1}}{\alpha + 1} dy + \int_0^1 \frac{(1 - y)^{\alpha+1}}{\alpha + 1} dy \\
 &= \left. \frac{y^{\alpha+2}}{(\alpha + 1)(\alpha + 2)} \right|_0^1 + \left. \frac{-(1 - y)^{\alpha+2}}{(\alpha + 1)(\alpha + 2)} \right|_0^1 \\
 &= \frac{1}{(\alpha + 1)(\alpha + 2)} + \frac{1}{(\alpha + 1)(\alpha + 2)} = \frac{2}{(\alpha + 1)(\alpha + 2)}
 \end{aligned}$$

Slightly easier is to observe that the first integral above is the integral over the triangle bounded by  $x = y$ ,  $x = 0$ ,  $y = 1$  and the second integral is what you get when you permute  $x$  and  $y$  in the first integral and so must have the same value, so you need only compute one integral and double the result.

7. P380:9. A total of  $n$  balls, numbered from 1 through  $n$ , are put into  $n$  urns, also numbered 1 through  $n$  in such a way that ball  $i$  is equally likely to go into any of the urns 1, 2,  $\dots$ ,  $i$ . Find:

- (a) The expected number of urns that are empty.

Let  $X_i$  be the Bernoulli r.v. which is 1 if urn  $i$  is empty, and 0 otherwise. Then  $X_i = 1 \iff$  for all  $j$  with  $i \leq j \leq n$ , ball  $j$  does not go into urn  $i$ . These  $n - i - 1$  events indexed by  $j$  are independent and  $P\{\text{ball } j \text{ does not go into urn } i\} = \frac{j - 1}{j}$ . Thus  $p_i = \prod_{j=i}^n \frac{j - 1}{j} = \frac{i - 1}{i} \frac{i}{i + 1} \dots \frac{n - 1}{n} = \frac{i - 1}{n}$ . Then

$$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = \sum_{i=1}^n \frac{i - 1}{n} = \frac{\sum_{i=1}^n (i - 1)}{n} = \frac{(n - 1)n/2}{n} = \frac{n - 1}{2}$$

- (b) The probability that none of the urns is empty.

There is no empty urn if and only if no urn has more than 1 ball since  $n$  is finite. Ball 1 goes into urn 1 with probability 1. Ball 2 goes into urn 1 or 2, and if there are not to be 2 balls in urn 1, it must go into urn 2 and the probability of this is  $\frac{1}{2}$ . Similarly, ball 3 must go into urn 3 and this has probability  $\frac{1}{3}$ ,  $\dots$ , and a simple induction shows ball  $i$  must go into urn  $i$ , which occurs with probability  $\frac{1}{i}$ . Then

$$P\{\text{No urn is empty}\} = \prod_{i=1}^n \frac{1}{i} = \frac{1}{n!}$$

8. p382:12. A group of  $n$  men and  $n$  women are lined up at random.

- (a) Find the expected number of men who have a woman next to them.

There are  $2n - 1$  spaces between two people on line. For  $1 \leq i \leq 2n - 1$ , let  $X_i$  be the Bernoulli r.v. which is 1 if the person in position  $i$  is male and the person in position  $i + 1$  is female, and 0 otherwise.

Then  $p_{X_i} = \frac{n^2}{(2n)(2n - 1)} = \frac{n}{2(2n - 1)}$ . For  $1 \leq i \leq 2n - 2$ , let  $Y_i$  be the Bernoulli r.v. which is 1 if there is a woman in position  $i$  and men in positions  $i + 1$  and  $i + 2$  and zero otherwise. Then  $p_{Y_i} = \frac{n(n(n - 1)/2)}{(2n)((2n - 1)(2n - 2)/2)} = \frac{n}{4(2n - 1)}$ . Let  $W$  be the Bernoulli r.v. which is 1 if the  $2n - 1^{\text{st}}$  person is female and the  $n^{\text{th}}$  person is male. Then  $p_W = \frac{n^2}{2n(2n - 1)}$ . The expected number of men with

a woman next to them is

$$E \left[ \sum_{i=1}^{2n-1} X_i + \sum_{i=1}^{2n-2} Y_i + W \right] = (2n-1) \frac{n}{2(2n-1)} + (2n-2) \frac{n}{4(2n-1)} + \frac{n}{2(2n-1)}$$

$$= \frac{3n^2 - n}{2(2n-1)}$$

- (b) If the group of  $2n$  people are seated at a round table, each man has two neighbors. The probability that both of these are male is  $\binom{n-1}{2} / \binom{2n-1}{2} = \frac{(n-1)(n-2)}{(2n-1)(2n-2)} = \frac{(n-2)}{2(2n-1)}$  so  $p_i = 1 - \frac{(n-2)}{2(2n-1)} = \frac{4n-2-n+2}{2(2n-1)} = \frac{3n}{2(2n-1)}$  and the expected number of men with a woman next to them is  $E[\sum_{i=1}^n X_i] = \sum_{i=1}^n E[X_i] = n \left( \frac{3n}{2(2n-1)} \right)$ .

9. P381:14 was done in class.

10. p385: 18 For each  $i$  between 1 and 52, let  $X_i$  be the Bernoulli r.v. which is 1 if and only if a card of face value  $i$  modulo 13 is put in position  $i$ . Then  $p_i = \frac{1}{13}$ . Then the expected number of matches is  $\sum_{i=1}^{52} X_i$  and its expected value is  $\sum_{i=1}^{52} E[X_i] = 4 \times 13 \times \frac{1}{13} = 4$ .

11. p382:21. Here we assume that there are 365 days in a year, and the distribution of birthdays is uniform across those 365 days. That is, leap years are just ignored, or the people born on February 29 are assigned randomly to other birthdays throughout the year, not just to February 28.

(a) Let  $X_i$  be the binomial random variable

$$X_i = \begin{cases} 1 & \text{precisely 3 people have their birthday on day } i \text{ of the year} \\ 0 & \text{otherwise} \end{cases}$$

Then  $E[X_i] = \binom{100}{3} \times 364^{97} / 365^{100} = \frac{100 \times 99 \times 98 \times 364^{97}}{6 \times 365^{100}} = 2.548342447 \times 10^{-3}$  and

$$E \left[ \sum_{i=1}^{365} X_i \right] = 365 \times 2.548342447 \times 10^{-3} = 0.9301449932$$

(b) Let  $Y_i$  be the binomial random variable

$$Y_i = \begin{cases} 1 & \text{nobody has a birthday on day } i \\ 0 & \text{otherwise} \end{cases}$$

$E[\sum Y_i] = 365 \times \frac{364^{100}}{365^{100}} = 277.4244819$  This is the expected number of days no one has a birthday, so the expected number of days someone does have a birthday is  $365 - 277.4244819 = 87.5755181$ .

There is another way to do this problem which I do not recommend without a computer algebra system (programming a calculator to do it will be more difficult than it is worth). Let  $Z_i$  be the Bernoulli random variable which is equal to 1 if at least one person has a birthday on day  $i$  and zero otherwise. That is,

$$Z_i = \begin{cases} 1 & \text{at least one person has a birthday on day } i \\ 0 & \text{otherwise} \end{cases}$$

Then  $p_{Z_i} = \sum_{n=1}^{100} \frac{100! \times 364^{(100-n)}}{n! \times (100-n)! \times 365^{100}}$  which is the sum of the probabilities that precisely  $n$  people have a birthday on day  $i$  as  $n$  goes from 1 to 100, so

$$E \left[ \sum_{i=1}^{365} Z_i \right] = \left( \sum_{n=1}^{100} \frac{100! \times 364^{(100-n)}}{n! \times (100-n)! \times 365^{100}} \right) \times 365 = 87.57551806$$

12. p383:30. If  $X$  and  $Y$  are independent and identically distributed with mean  $\mu$  and variance  $\sigma^2$ , find  $E[(X - Y)^2]$ .

$$\begin{aligned} E[(X - Y)^2] &= E[X^2 - 2XY + Y^2] \\ &= E[X^2] - 2E[X]E[Y] + E[Y^2] \\ &= 2(\sigma^2 + \mu^2) - 2\mu^2 = 2\sigma^2 \end{aligned}$$

13. P383:31. A fair die is rolled 10 times. Let  $X$  be the sum of results  $\{X_i : 1 \leq i \leq 10\}$  of these rolls. Each roll generates a uniform random variable  $X_i$  on  $\{1, 2, 3, 4, 5, 6\}$  and so has mean  $\mu = \frac{6 \times 7/2}{6} = \frac{7}{2}$  and  $E[X_i^2] = \frac{1 + 4 + 9 + 16 + 25 + 36}{6} = \frac{91}{6}$  so  $\text{var}(X_i) = \frac{91}{6} - \left(\frac{7}{2}\right)^2 = \frac{35}{12}$ . Then

$$\sigma^2 = \text{var}\left(\sum_{i=1}^{10} X_i\right) = \sum_{i=1}^{10} \text{var}(X_i) = \frac{350}{12} = \frac{175}{6}$$