The computations here were done in the process of creating the file for this answer sheet, and that is a prescription for errors to creep in. Please report any to me. I may also change the file before 5/3 to add problems or correct errors I find, but I wanted to get this up as soon as possible.

1. p. 383:26. Let \( \{X_i : 1 \leq i \leq n\} \) be independent random variables all having uniform distributions over \((0, 1)\).

   (a) If \( Y = \max (X_i) \), then \( F_Y (a) = P \{X_i \leq a \text{ for all } i\} = a^n \) so \( f_Y (a) = na^{n-1} \) and

   \[
   E[\max (X_i)] = \int_0^1 y (ny^{n-1}) \, dy = \int_0^1 ny^n \, dy = \frac{n}{n+1}
   \]

   (b) If \( Y = \min (X_i) \), then \( P \{Y > a\} = P \{X_i > a \text{ for all } i\} = (1-a)^n = 1-F_Y (a) \), so we differentiate both sides to get \(-n(1-a)^{n-1} = -f_Y (a) \) and

   \[
   E[\min (X_i)] = \int_0^1 ny (1-y)^{n-1} \, dy
   \]

   \[
   = ny \left[ \frac{(1-y)^n}{n} \right]_0^1 + n \int_0^1 \frac{(1-y)^n}{n} \, dy
   \]

   \[
   = -\left( \frac{(1-y)^{n+1}}{n+1} \right)_0^1 = \frac{1}{n+1}
   \]

2. p. 384:34. If 10 married couples are randomly seated at a round table, compute the expected number and the variance of the number of wives who are seated next to their husbands.

Number the couples from 1 to 10 and let \( X_i \) be the Bernoulli r.v. that is 1 precisely when couple \( i \) are seated together. Then the number \( X \) of wives who are seated next to their husbands is

\[
X = \sum_{i=1}^{10} X_i
\]

(a) \( E[X] = 10p \), where \( p \) is the probability that couple \( i \) is seated together. Consider the seating as first seating wife \( i \). If couple \( i \) are seated together, then there are 2 places to seat husband \( i \) and 18! ways of seating the rest of the people. If there is no such restriction, there are 19! ways of seating the other 19 people. Hence \( p = \frac{219!}{19!} = \frac{2}{19} \), so

\[
E[X] = \frac{20}{19}
\]

(b) The major problem here is that the \( X_i \) are not independent.

\[
\text{Var}(X) = \text{Cov} \left( \sum_{i=1}^{10} X_i, \sum_{j=1}^{10} X_j \right) = \sum_{i=1}^{10} \sum_{j=1}^{10} \text{Cov}(X_i, X_j)
\]

\[
= \sum_{i=1}^{10} \text{Cov}(X_i, X_i) + \sum_{i=1}^{10} \sum_{j=1, j \neq i}^{10} \text{Cov}(X_i, X_j)
\]

\[
= 10 \cdot \frac{2}{19} \cdot \frac{17}{19} + 10 \cdot 9 \cdot \text{Cov}(X_1, X_2)
\]

since all of the \( \text{Cov}(X_i, X_j) \) with \( i \neq j \) are equal.

\[
\text{Cov}(X_1, X_2) = E[X_1X_2] - E[X_1]E[X_2] = E[X_1X_2] - \frac{4}{19^2}
\]

\[
= P \{\text{Couple 1 is seated together and couple 2 is seated together}\} - \frac{4}{19^2}
\]
Seat wife 1. There are then 2 ways that you can seat her husband next to her, and 2 \cdot 17! ways to seat the remaining 18 people so couple 2 is seated together. Thus

$$P \{ \text{Couple 1 is seated together and couple 2 is seated together} \} = \frac{4 \cdot 17!}{19!}.$$ 

Then

$$\text{Cov} (X_1, X_2) = \frac{4}{19 \cdot 18} - \frac{4}{19^2} = \frac{4 \cdot 19 - 4 \cdot 18}{18 \cdot 19^2} = \frac{2}{9 \cdot 19^2}$$

and

$$\text{Var} (X) = 10 \cdot \frac{2}{19} \cdot \frac{17}{19} + 10 \cdot \text{Cov} (X_1, X_2)$$

$$= 10 \cdot \frac{2}{19} \cdot \frac{17}{19} + 10 \cdot \left( \frac{2}{9 \cdot 19^2} \right)$$

$$\frac{340 + 20}{19^2} = \frac{360}{361}.$$

3. p384.36. Let $X$ be the number of 1’s and $Y$ the number of 2’s that occur in $n$ rolls of a fair die. Compute $\text{Cov} (X, Y)$.

Let $U_i$ be the Bernoulli r.v. which is 1 when roll $i$ is a 1 and 0 otherwise, and let $V_j$ be the Bernoulli r.v. which is 1 when roll $i$ is a 2 and 0 otherwise. Then $X = \sum_{i=1}^{n} U_i$ and $Y = \sum_{j=1}^{n} V_j$ and for any $i$ and $j$ with $i \neq j$ $\text{Cov} (U_i, V_j) = 0$ because they are independent, so we have

$$\text{Cov} (X, Y) = \text{Cov} \left( \sum_{i=1}^{n} U_i , \sum_{j=1}^{n} V_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov} (U_i, V_j)$$

$$= \sum_{i=1}^{n} \text{Cov} (U_i, V_i) = n \left( 0 - E \left[ U_1 \right] E \left[ V_1 \right] \right) = -\frac{n}{36}.$$

4. p386.40. The joint density function of $X$ and $Y$ is given by

$$f(x, y) = \frac{1}{y} \exp \left( - \left( y + \frac{x}{y} \right) \right) = \frac{e^{-y}}{y} e^{-x/y} \quad x > 0, \quad y > 0$$

Find $E[X]$, $E[Y]$, and $\text{Cov} (X, Y)$.

We first check check that this is a probability distribution by looking at $\int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{y} e^{-y} e^{-x/y} \, dx \, dy = \int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{y} e^{-y} e^{-x/y} \, dy \, dx$. The region of integration is the first quadrant, so it will not influence order. If one integrates with respect to $y$ first, one is faced with integrating a (function of $y$) $\times \exp (x/y)$ with $x$ held constant and the derivative of $1/y$ no where in sight. This is not a closed form integral, so we will try the other order for this, and all of the other integrals. $\int x^n e^{ax} \, dx$ can be found by integrating by parts or using a table of integrals or using a CAS such as Maple. We also use the fact that any polynomial times $e^{-ax}$ goes to 0 as $x$ goes to $\infty$ if $a > 0$.

$$\int_{0}^{\infty} \int_{0}^{\infty} \frac{1}{y} e^{-y} e^{-x/y} \, dx \, dy = \int_{y=0}^{\infty} e^{-y} \left[ \int_{x=0}^{\infty} e^{-x/y} \, dx \right] dy$$

$$= \int_{0}^{\infty} \frac{e^{-y}}{y} \left( -y \, e^{-x/y} \right) \bigg|_{x=0}^{\infty} \, dy = \int_{0}^{\infty} e^{-y} \, dy = 1$$

(a)

$$E[X] = \int_{0}^{\infty} \int_{0}^{\infty} \frac{x}{y} e^{-y} e^{-x/y} \, dx \, dy = \int_{0}^{\infty} \frac{e^{-y}}{y} \left[ \int_{0}^{\infty} xe^{-x/y} \, dx \right] dy$$

$$= \int_{x=0}^{\infty} e^{-y} \left( -xy \, e^{-x/y} - y^2 \, e^{-x/y} \right) \bigg|_{x=0}^{\infty} \, dy = \int_{0}^{\infty} \frac{e^{-y}}{y} \left( y^2 \right) \, dy$$

$$= \left( -y \, e^{-y} - e^{-y} \right) \bigg|_{y=0}^{\infty} = 1$$
(b) 
\[ E[Y] = \int_0^\infty \int_0^\infty e^{-y} e^{-x/y} \, dx \, dy = \int_0^\infty e^{-y} \left[ \int_0^\infty e^{-x/y} \, dx \right] \, dy \\
= \int_{x=0}^{x=\infty} e^{-y} \left( -y e^{-x/y} \right) \bigg|_{x=0}^{x=\infty} \, dy = \int_0^\infty y e^{-y} \, dy \\
= (-y e^{-y} - e^{-y}) \bigg|_{y=0}^{y=\infty} = 1 \]

(c) 
\[ E[XY] = \int_0^\infty \int_0^\infty xe^{-x/y} e^{-x/y} \, dx \, dy = \int_0^\infty e^{-y} \left[ \int_0^\infty xe^{-x/y} \, dx \right] \, dy \\
= E[Y] = 1 \]
\[ \text{Cov}(X, Y) = E[XY] - E[X]E[Y] = 1 - 1 = 0 \]

so \( \text{Cov}(X, Y) = 2 - 1 = 1 \).

5. p385:45. Let \( \{X_i : 1 \leq i \leq 4\} \) be independent pairwise uncorrelated random variables each having mean 0 and variance 1, each having mean 0 and variance 1.

Since these r.v.’s are uncorrelated, \( \rho(X_i , X_j) = \frac{\text{Cov}(X_i , X_j)}{\sqrt{\text{Var}(X_i) \cdot \text{Var}(X_j)}} = 0 \) if \( i \neq j \). In particular, \( \text{Cov}(X_i , X_j) = 0 \) if \( i \neq j \).

(a) 
\[ \rho(X_1 + X_2 , X_2 + X_3) = \frac{\text{Cov}(X_1 + X_2 , X_2 + X_3)}{\sqrt{\text{Var}(X_1 + X_2) \cdot \text{Var}(X_2 + X_3)}} \\
= \frac{\text{Cov}(X_1, X_2) + \text{Cov}(X_1, X_3) + \text{Cov}(X_2, X_2) + \text{Cov}(X_2, X_3)}{\sqrt{(\text{Var}(X_1) + \text{Var}(X_2) + 2 \text{Cov}(X_1, X_2))(\text{Var}(X_2) + \text{Var}(X_3) + 2 \text{Cov}(X_2, X_3))}} \\
= \frac{0 + 0 + 1 + 0}{\sqrt{(1 + 1 + 0)(1 + 1 + 0)}} = \frac{1}{\sqrt{4}} \cdot \frac{1}{2} = \frac{1}{2} \]

(b) \( \text{Cov}(X_1 + X_2 , X_3 + X_4) = 0 \) since the two sums are independent.

6. p385:48. A fair die is successively rolled. Let \( X \) be the number of rolls necessary to obtain a 6, and \( Y \) the number of rolls necessary to obtain a 5. Find:

(a) \( E[X] \). \( X \) is a geometric random variable with parameter \( \frac{1}{6} \), so \( E[X] = \frac{1}{1/6} = 6 \).

(b) \( E[X \mid Y = 1] \). If you know that the first roll is not a 6, the expected number of rolls after this first one is 6, so the total expected number of rolls given that the first is a 5 is \( 6 + 1 = 7 \).

(c) \( E[X \mid Y = 5] \).

\[ P_{X \mid Y}(X = i \mid Y = 5) = \begin{cases} 
\left( \frac{4}{6} \right)^2 \left( \frac{5}{6} \right)^{i-1} \left( \frac{1}{6} \right)^4 / \left( \frac{6}{5} \right)^4 \left( \frac{1}{6} \right)^4 & \text{if } 0 \leq i \leq 4 \\
0 & \text{if } i = 5 \\
\left( \frac{4}{6} \right)^4 \left( \frac{1}{6} \right)^{i-6} / \left( \frac{5}{6} \right)^4 \left( \frac{1}{6} \right)^4 = \left( \frac{4}{6} \right)^4 \left( \frac{1}{6} \right)^{i-6} \left( \frac{5}{6} \right) & \text{if } 6 \leq i 
\end{cases} \]

so

\[ E[X \mid Y = 5] = 1 \cdot \frac{1}{5} + 2 \cdot \frac{4}{5} + 3 \cdot \frac{4^2}{5^2} + 4 \cdot \frac{4^3}{5^3} + 5 \cdot \frac{4^4}{6 \cdot 5^4} \sum_{i=6}^{\infty} i \cdot \left( \frac{5}{6} \right)^{i-6} \\
= 1.3136 + \frac{4^4}{6 \cdot 5^4} \left( \frac{6}{5} \right)^5 \sum_{i=6}^{\infty} i \cdot \left( \frac{5}{6} \right)^{i-1} - \frac{4^4}{6 \cdot 5^4} \left( \frac{6}{5} \right)^5 \sum_{i=1}^{5} i \cdot \left( \frac{5}{6} \right)^{i-1} \\
= 1.3136 + \frac{4^4}{6 \cdot 5^4} \left( \frac{6}{5} \right)^5 \left( \frac{1}{1 - \frac{5}{6}} \right)^2 - \frac{4^4}{6 \cdot 5^4} \left( \frac{6}{5} \right)^5 \sum_{i=1}^{5} i \cdot \left( \frac{5}{6} \right)^{i-1} = 5.8192 \]
7. p385:50. The joint density of \( X \) and \( Y \) is given by

\[
f(x, y) = \frac{e^{-x/y} e^{-y}}{y} \quad 0 < x < \infty, \quad 0 < y < \infty
\]

Compute \( E[X^2 \mid Y = y] \).

\[
P\{Y = y\} = \int_{x=0}^{x=\infty} \frac{e^{-x/y} e^{-y}}{y} \, dx = - (e^{-x/y})|_{x=0}^{x=\infty} = e^{-y} \text{ so } f_{X \mid Y}(x \mid Y = y) = \frac{e^{-x/y}}{y}. \]

Then

\[
E[X^2 \mid Y = y] = \int_{x=0}^{x=\infty} x^2 \frac{e^{-x/y}}{y} \, dx
\]

\[
= \left( \frac{1}{y} \right) \left( -2y^3 e^{-y} - 2xy^2 e^{-y} - x^2 ye^{-y} \right) \bigg|_{x=0}^{x=\infty} = 2y^2
\]

8. p427:1. Let \( X \) be a r.v. with \( \mu = \sigma^2 = 20 \). What can you say about \( P\{0 < X < 40\} \).

Let \( Y = \lfloor X - 20 \rfloor \), and note that \( \sigma^2 = \text{Var}(Y) \).

\[
P\{0 < X < 40\} = P\{|Y - 20| < 20\};
\]

\[
= \frac{20}{20^2} = P\{|Y - 20| \geq 20\} \quad \text{(by Chebyshev)}
\]

\[
= 1 - P\{|Y - 20| < 20\}
\]

so \( P\{0 < X < 40\} \geq 1 - \frac{1}{20} \).

9. p427:2. From past experience a professor knows that the text score of a student taking his or her final exam is a random variable with mean 75. In addition, in the last two parts of the question suppose that the variance is 25.

(a) Give an upper bound for the probability that a student’s test score will exceed 85. Let \( X \) denote the student’s score. Then Markov’s inequality gives

\[
P\{X \geq 85\} \leq \frac{75}{85} = \frac{15}{17}
\]

(b) What can be said about the probability that a student will score between 65 and 85. By Chebyshev’s inequality,

\[
P\{|X - 75| \geq 10\} \leq \frac{25}{100} = \frac{1}{4}
\]

so \( P\{|X - 75| \leq 10\} \geq 1 - \frac{1}{4} = \frac{3}{4} \).

(c) How many students would have to take the test to ensure, with probability at least .9 that the class average would be within 5 of 75. By the proof of the weak law of large numbers using Chebyshev,

\[
P\left\{ \left| \frac{\sum_{i=1}^{n} X_i}{n} - 75 \right| \geq 5 \right\} \leq \frac{25}{25n} = \frac{1}{n}
\]

so \( P\left\{ \left| \frac{\sum_{i=1}^{n} X_i}{n} - 75 \right| \leq 5 \right\} \leq 1 - \frac{1}{n} \) and this is \( \leq .9 \iff n \geq 10 \).

10. p427:3. Redo the last part of the previous question using the central limit theorem. The central limit theorem states that if \( n \) is large enough and the \( X_i \) are independent, identically distributed random variables

\[
P\left\{ \frac{\sum_{i=1}^{n} X_i - 75n}{\sqrt{n}} \leq a \right\} \approx \Phi(a)
\]

so

\[
P\left\{ \left| \frac{\sum_{i=1}^{n} X_i - 75n}{\sqrt{n}} \right| \leq a \right\} \approx 2\Phi(a) - 1.
\]
2\Phi(a) - 1 = 0.9 \iff \Phi(a) = \frac{0.9}{2} = 0.95. From tables, a = 1.645 (this one turns out to be trivial to interpolate).

\[
\left| \frac{\sum_{i=1}^{n} X_i}{n} - 75 \right| \leq 5 \iff \left| \sum_{i=1}^{n} X_i - 75n \right| \leq 5n \iff \left| \sum_{i=1}^{n} X_i - 75n \right| \leq 5n \implies \sqrt{n} \leq \sqrt{n}
\]

so we need to find the smallest \( n \) such that \( \sqrt{n} \geq 1.645 \). The smallest such \( n \) is 3.

11. p427:5. Fifty numbers are rounded off to the nearest integer and then summed. If the individual resultant round-off errors are uniformly distributed over \((-0.5, 0.5)\), what is the probability that the resultant sum differs from the exact sum by more than 3.

Let \( X_i \) be the round-off error in number \( i \). Then \( E[X_i] = 0 \) and \( \text{Var}(X_i) = \frac{1}{12} \). By Chebyshev’s inequality, \( P\left\{ \left| \sum_{i=1}^{50} X_i \right| \geq 3 \right\} \leq \frac{50}{108} = 0.46296 \) but that is not a good estimate at all. We estimate using the central limit theorem.

\[
P\left\{ \frac{\sum_{i=1}^{50} X_i - 0}{\sqrt{\frac{50}{12}}} > \frac{3}{\sqrt{\frac{50}{12}}} \approx 1.47 \right\} \approx 1 - \Phi(1.47) = 1 - 0.9292 = 0.0708 \quad \text{so}
\]

\[
P\left\{ \left| \sum_{i=1}^{50} X_i \right| > 3 \right\} \approx 2(0.0708) = 0.1416
\]

12. p427:14. Let \( X_i \) be the lifetime of the \( i^{th} \) component. Then \( \mu = 100 \) and \( \sigma = 30 \). You are asked to find an \( n \) so that \( P\{\sum_{i=1}^{n} X_i < 2000\} < 0.05 \). By the central limit theorem, this is approximately the probability that a normal r.v. with mean 100\( n \) and \( \sigma = 30\sqrt{n} \) is less than 2000.

\[
P\left\{ \frac{\sum_{i=1}^{n} X_i - 100n}{30\sqrt{n}} < \frac{2000 - 100n}{30\sqrt{n}} \right\} \approx \Phi\left( \frac{2000 - 100n}{30\sqrt{n}} \right)
\]

will be less than 0.05 if \( \Phi\left( \frac{2000 - 100n}{30\sqrt{n}} \right) < 0.05 \). This is the case when \( \Phi\left( \frac{2000 - 100n}{30\sqrt{n}} \right) > 0.95 \) so

\[
-\frac{2000 - 100n}{30\sqrt{n}} > 1.645, \text{ or } 100(\sqrt{n})^2 - (30)(1.645)\sqrt{n} - 2000 > 0. \text{ This polynomial in } \sqrt{n} \text{ factors as}
\]

\[
100(\sqrt{n})^2 - (30)(1.645)\sqrt{n} - 2000 = 100(\sqrt{n} + 4.232)(\sqrt{n} - 4.726)
\]

For this polynomial to be positive, \( \sqrt{n} \) must be > 4.7257, so take \( n > (4.7257)^2 \approx 22.3 \). That is, you must have at least 23 components on hand to have .95 probability of operating more than 2000 hours.