Solution to review problem 5

5) Random samples of size 10 from two independent normal populations, each having variance 5, yield $\overline{x}_1 = 4$ and $\overline{x}_2 = 4.9$.
(a) Test the null hypothesis $\mu_1 \geq \mu_2$ at the 5% level.
(b) If in fact $\mu_1 = \mu_2 - 1$, what is the probability of a type II error with the test you used in (a)?

More generally, consider any two independent normal populations having the same (known) variance σ, and having unknown means μ_1 and μ_2. Let $x_1, \ldots, x_n$ be a random sample from the first population, and let $y_1, \ldots, y_n$ be a random sample from the second population (I am assuming that these samples have the same size n). We are testing the null hypothesis $H_0 : \mu_1 \geq \mu_2$ against the alternative hypothesis $H_1 : \mu_1 < \mu_2$. Since this null hypothesis is composite, the results from chapters 8 and 11 don't give a way to test it. The only method we have for testing composite null hypotheses is the likelihood ratio test. The critical regions for this test have the form $\lambda < k$, where $0 < k < 1$ and where λ is the likelihood ratio statistic (which is a function of the x_i's and y_j's). So we must compute λ.

To do this, first write out the joint density function for $x_1, \ldots, x_n, y_1, \ldots, y_n$: since as usual the x_i's and y_j's are independent, this is

$$\begin{aligned} f(x_1, \ldots, x_n, y_1 \ldots, y_n; \mu_1, \mu_2) &= \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i - \mu_1}{\sigma}\right)^2} \prod_{j=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{y_j - \mu_2}{\sigma}\right)^2} \\ &= \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{2n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left((x_i-\mu_1)^2 + (y_i-\mu_2)^2\right)}. \end{aligned}$$

Recall that

$$\lambda = \frac{\max_{(\mu_1,\mu_2)\in H_0} f(x_1, \ldots, x_n, y_1, \ldots, y_n; \mu_1, \mu_2)}{\max_{(\mu_1,\mu_2)\in H_0 \cup H_1} f(x_1, \ldots, x_n, y_1, \ldots, y_n; \mu_1, \mu_2)}.$$

We're viewing all the x_i's and y_j's as being fixed, so for simplicity write

$$L(\mu_1, \mu_2) = f(x_1, \ldots, x_n, y_1, \ldots, y_n; \mu_1, \mu_2).$$

First let's compute the maximum that occurs in the denominator of λ, so we're maximizing $L(\mu_1, \mu_2)$ over all real numbers μ_1, μ_2. The maximum occurs for the same pair (μ_1, μ_2) which maximizes

$$\log L(\mu_1, \mu_2) = -2n\log(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left((x_i-\mu_1)^2 + (y_i-\mu_2)^2\right).$$

Taking the partial dervative with respect to μ_1, we find that $\log L(\mu_1, \mu_2)$ is maximized (for fixed μ_2) when $\mu_1 = \overline{x}$; likewise, for fixed μ_1 the maximum occurs when $\mu_2 = \overline{y}$. Thus the denominator of λ is just $L(\overline{x}, \overline{y})$.

Now consider the numerator of λ, so we're maximizing $L(\mu_1, \mu_2)$ over all numbers μ_1, μ_2 such that $\mu_1 \geq \mu_2$. First let's check that the maximum over *all* μ_1, μ_2 doesn't occur in this region, i.e., that the values of μ_1 and μ_2 which give rise to this maximum should not satisfy $\mu_1 \geq \mu_2$ (if they satisfied this, then the numerator and denominator of λ would be the same, so $\lambda = 1$ and this test won't be of any use). Well, the maximum over *all* μ_1, μ_2 occurs when $\mu_1 = \overline{x}$ and $\mu_2 = \overline{y}$, and in our specific example indeed $\overline{x} = 4 < 4.9 = \overline{y}$. So that's good.

In order to maximize $L(\mu_1, \mu_2)$ over the region $\mu_1 \geq \mu_2$, let's introduce a new variable $\nu = \mu_1 - \mu_2$, so $\mu_1 = \nu + \mu_2$; then our region becomes just $\nu \geq 0$, which is easier to visualize. Rewrite $\log L(\mu_1, \mu_2)$ in terms of ν and μ_2:

$$\log L(\mu_1, \mu_2) = -2n\log(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left((x_i-\nu-\mu_2)^2 + (y_i-\mu_2)^2\right).$$

We want to maximize this number (call it g) over all μ_2 and all $\nu \geq 0$. First set $\frac{\partial g}{\partial \mu_2}$ to zero, and find that g is maximized (for fixed ν) when $\mu_2 = (\overline{x} + \overline{y} - \nu)/2$. Now substitute this value of μ_2 into the expression for g, to get

$$g = -2n\log(\sigma\sqrt{2\pi}) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(\left(x_i - \frac{\overline{x}+\overline{y}+\nu}{2}\right)^2 + \left(y_i - \frac{\overline{x}+\overline{y}-\nu}{2}\right)^2\right).$$

Now we need to maximize this last expression over all $\nu \geq 0$. Compute

$$\begin{aligned}
\frac{\partial g}{\partial \nu} &= -\frac{1}{2\sigma^2}\sum_{i=1}^{n}\left(-\left(x_i - \frac{\overline{x}+\overline{y}+\nu}{2}\right) + \left(y_i - \frac{\overline{x}+\overline{y}-\nu}{2}\right)\right) \\
&= -\frac{1}{2\sigma^2}\left(-\left(n\overline{x} - \frac{n}{2}(\overline{x}+\overline{y}+\nu)\right) + \left(n\overline{y} - \frac{n}{2}(\overline{x}+\overline{y}-\nu)\right)\right) \\
&= \frac{n}{2\sigma^2}\left(\overline{x} - \overline{y} - \nu\right),
\end{aligned}$$

which is negative because $\overline{x} - \overline{y} = 4 - 4.9 < 0$ and $-\nu \leq 0$. Thus, g is a decreasing function of ν when $\nu \geq 0$, so g is maximized (in this range) when $\nu = 0$. Recalling that $\mu_2 = (\overline{x} + \overline{y} - \nu)/2$ and $\mu_1 = \nu + \mu_2$, this says the numerator of λ is $L\left(\frac{\overline{x}+\overline{y}}{2}, \frac{\overline{x}+\overline{y}}{2}\right)$.

Now we've computed λ, and we just need to simplify the resulting expression. We have

$$\begin{aligned}\lambda &= \frac{L\left(\frac{\overline{x}+\overline{y}}{2}, \frac{\overline{x}+\overline{y}}{2}\right)}{L(\overline{x}, \overline{y})} \\ &= e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n\left(\left(x_i - \frac{\overline{x}+\overline{y}}{2}\right)^2 + \left(y_i - \frac{\overline{x}+\overline{y}}{2}\right)^2 - (x_i - \overline{x})^2 - (y_i - \overline{y})^2\right)},\end{aligned}$$

and by writing $\left(x_i - \frac{\overline{x}+\overline{y}}{2}\right)^2 = \left((x_i - \overline{x}) + \frac{\overline{x}-\overline{y}}{2}\right)^2$ and expanding, we get

$$\lambda = e^{-\frac{(\overline{x}-\overline{y})^2}{4\sigma^2}}.$$

The critical regions are $\lambda \leq k$ where $0 < k < 1$; equivalently (taking logs), these are

$$\frac{-(\overline{x} - \overline{y})^2}{4\sigma^2} < \log k,$$

or equivalently

$$(\overline{x} - \overline{y})^2 > -4\sigma^2 \log k.$$

In the computation of λ, we assumed that $\overline{x} < \overline{y}$, so the critical region can be rewritten as

$$\overline{y} - \overline{x} > \sigma\sqrt{-\log k},$$

or equivalently $\overline{y} - \overline{x} > c$ for some constant $c > 0$.

Now we must determine how the size α of the critical region depends on c. For this we can use the result of exercise 8.3 (which I believe I proved in class), namely that $\overline{y} - \overline{x}$ has normal distribution with mean $\mu_2 - \mu_1$ and variance $2\sigma^2/n$. Then

$$z := \frac{\overline{y} - \overline{x} - (\mu_2 - \mu_1)}{\sigma\sqrt{2/n}}$$

has standard normal distribution, so by definition

$$P(z > z_\alpha) = \alpha,$$

and thus

$$P(\overline{y} - \overline{x} > \mu_2 - \mu_1 + z_\alpha\sigma\sqrt{2/n}) = \alpha.$$

But this depends on $\mu_2 - \mu_1$, and we need a critical region that only depends on the x_i's and y_j's. Note that, for any specific constant $\delta \leq 0$, we have computed the critical region of size α for testing the same alternative hypothesis against the more specific null hypothesis $\mu_2 - \mu_1 = \delta$. To find a critical region that simultaneously works for all values of δ, we take the intersection of all corresponding critical regions

$$\overline{y} - \overline{x} > \delta + z_\alpha \sigma \sqrt{2/n},$$

getting

$$\overline{y} - \overline{x} > z_\alpha \sigma \sqrt{2/n}.$$

Since we took the intersection of the different critical regions, it follows that when the null hypothesis holds, the probability that $\overline{y}$ and $\overline{x}$ satisfy the above inequality if at most α; and moreover, if actually $\delta = 0$, or equivalently $\mu_2 = \mu_1$, then the probability equals α.

We have shown that the critical region of size α produced by the likelihood ratio test is

$$\overline{y} - \overline{x} > z_\alpha \sigma \sqrt{2/n}.$$

Now consider the specific problems that were asked. There $n = 10$ and $\sigma^2 = 5$, so the critical region of size 0.05 is

$$\overline{y} - \overline{x} > z_{.05} \sqrt{5} \sqrt{2/10} = z_{.05} = 1.65.$$

This critical region does not include the specific pair $(\overline{x}, \overline{y}) = (4, 4.9)$, so we must accept the null hypothesis.

Next suppose $\mu_1 = \mu_2 - 1$. Then the probability of a type II error is

$$P(\overline{y} - \overline{x} < 1.65).$$

As noted above,

$$z := \frac{\overline{y} - \overline{x} - (\mu_2 - \mu_1)}{\sigma \sqrt{2/n}}$$

has standard normal distribution, and here $\mu_2 - \mu_1 = 1$ and $\sigma \sqrt{2/n} = 1$, so $z = \overline{y} - \overline{x} - 1$. Thus

$$P(\overline{y} - \overline{x} < 1.65) = P(z < 0.65) = 0.7422.$$