MATH 640:491–Fall, 2006

Suggested problems for 9/19/06

1. Let S be a square of side length 1, and let $P_1, P_2, \ldots, P_5$ be five points in the interior opf the square. Prove that one can find two of these five points such that the distance between them is at most $\sqrt{2}$.

2. Let Q_1, Q_2, Q_3, Q_4, Q_5 be 5 points in the x-y plane having integer cooredinates. Prove that there is a pair of points among them whose midpoint has integer coordinates.

3. Let A be a subset of integers of size n. Prove that there is a subset of A whose sum is divisible by n.

4. Let A be a subset of size $n+1$ consisting of positive integers in the range 1 to $2n$. Prove that there must be distinct elements a, b of A such that a is a divisor of b

5. Let A be a subset of size $n+1$ consisting of positive integers in the range 1 to $2n$. Prove that there must be elements x, y, z (not necessarily distinct) of A such that $x + y = z$.

6. Let A be a finite subset of positive integers of size n Let $a_1, a_2, \ldots, a_t$ be a sequence of integers each belonging to A. Prove that if $t \geq 2^n$ then there are integers j, k satisfying $1 \leq j \leq k \leq n$ such that $\prod_{i=j}^{k} a_i$ is a perfect square.

7. Let s, t be positive integers. Let $x_1, \ldots, x_s$ be positive integers with each at most t and $y_1, \ldots, y_t$ be positive integers with each at most s. Prove that there is a subset I of $\{1, \ldots, s\}$ and a subset J of $\{1, \ldots, t\}$ such that $\sum_{i \in I} x_i = \sum_{j \in J} y_j$.

8. Let M be a matrix of real numbers, with each row in nondecreasing order. Suppose we sort each column into nondecreasing order. Prove that the rows are still in nondecreasing order.

9. (Somewhat difficult) Let A, B be integer 2 by 2 matrices. Suppose that each of the matrices $A, A + B, A + 2B, A + 3B, A + 4B$ has the property that it is invertible and its inverse has integer entries. Prove that $A + 5B$ has the same property.