MATH 640:491–Fall, 2006

Suggested problems for 10/31/06[1]

1. Let P be a non-constant polynomial given by:

$$P(x) = a_n x^n + \cdots + a_1 x + a_0.$$

Assume that P has n distinct nonzero roots $r_1, \ldots, r_n$. Prove that:

$$\frac{1}{r_1} + \cdots + \frac{1}{r_n} = -\frac{a_1}{a_0}.$$

2. For k a nonnegative integer and x a variable define:

$$\binom{x}{k} = \frac{x(x-1)\ldots(x-k+1)}{k!}.$$

Let $P(x)$ be a real polynomial of degree n.

   (a) Prove that there are unique reals $a_0, a_1, \ldots, a_n$ such that $P(x) = a_0\binom{x}{0} + a_1\binom{x}{1} + \cdots a_n\binom{x}{n}$.

   (b) Prove that the coefficients $a_0, \ldots, a_n$ in the first part are all integers if and only if P maps integers to integers.

3. Prove that if the quadratics $ax^2 + bx + c$ and $px^2 + qx + r$ have a common root then $(ar - cp)^2 = (aq - bp)(br - cq)$.

4. Prove that if P and Q are polynomials with $P^2 - Q^3 = 1$ then P and Q are constant polynomials.

5. Let $P(x)$ be a polynomial of degree $n \geq 1$ and distinct roots $r_1, \ldots, r_n$. Prove that for any number a such that $P'(a) \neq 0$ there is an $i \in \{1, \ldots, n\}$ such that $|a - r_i| \leq n|P(a)|/|P'(a)|$.

6. (Putnam 1985, B2) Define polynomials $f_n(x)$ for $n \geq 0$ by $f_0(x) = 1$, $f_n(0) = 0$ for $n \geq 1$ and $f'_{n+1}(x) = (n+1)f_n(x+1)$ for $n \geq 0$. Find with proof, the explicit factorization of $f_{100}(1)$ into powers of distinct primes.

7. (Putnam 1986, A6) Let $a_1, \ldots, a_n$ be real numbers and $b_1, \ldots, b_n$ be distinct positive integers. Suppose there is a polynomial $f(x)$ satisfying:

$$(1-x)^n f(x) = 1 + \sum_{i=1}^{n} a_i x^{b_i}.$$

Find a simple expression (not involving summations) for $f(1)$ in terms of $b_1, \ldots, b_n$ and n (but independent of $a_1, \ldots, a_n$).

[1]Version:10/24/06

8. (Putnam 1986, B5). Let $f(x, y, z) = x^2+y^2+z^2+xyz$. Let $p(x, y, z), q(x, y, z), r(x, y, z)$ be real polynomials satisfying:

$$f(p(x, y, z), q(x, y, z), r(x, y, z)) = f(x, y, z).$$

Prove or disprove the assertion that the sequence p, q, r consists of some permutation of $\pm x, \pm y, \pm z$ where the number of minus signs is 0 or 2.

9. (Putnam 1991, B5). Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial $p_n(x) = a_0 + a_1x + \cdots + a_nx^n$ has exactly n distinct real roots?