MA 491 problem set for November 7, 2006

Inequalities

1. If a, b, c are positive numbers such that $a+b+c=1$, prove that $ab+bc+ac \leq 1/3$.

2. If $a_1/b_1,\ a_2/b_2, \ldots, a_n/b_n$ are n fractions with $b_i > 0$ for all i, show that the fraction

$$\frac{a_1 + \ldots + a_n}{b_1 + \ldots + b_n}$$

lies between the largest and smallest of these fractions.

3. If a and b are nonzero real numbers, prove that at least one of the following inequalities holds:

$$\left| \frac{a + \sqrt{a^2 + 2b^2}}{2b} \right| < 1, \qquad \left| \frac{a - \sqrt{a^2 + 2b^2}}{2b} \right| < 1.$$

4. If the n numbers x_1, x_2, $\ldots, x_n$ lie in the interval $(0,\,1)$, prove that at least one of the following inequalities holds:

$$x_1 x_2 \ldots x_n \leq 2^{-n}, \qquad (1-x_1)(1-x_2)\ldots(1-x_n) \leq 2^{-n}.$$

5. Prove that for all n,

$$\left(\frac{n}{e}\right)^n < n! < e\left(\frac{n}{2}\right)^n$$

6. Prove that for any positive integer n, $\sqrt[n]{n} < 1 + \sqrt{2/n}$.

7. Suppose that $a_1, \ldots, a_n$ are positive numbers and $b_1, \ldots, b_n$ is a rearrangement of $a_1, \ldots, a_n$. Show that

$$\frac{a_1}{b_1} + \ldots + \frac{a_n}{b_n} \geq n.$$

8. For each integer $n > 2$, prove that $n! < \left(\dfrac{n+1}{2}\right)^n$

9. For n a positive integer, let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be permutations of $1, \ldots, n$. Find sharp lower and upper bounds for $a_1 b_1 + \ldots + a_n b_n$.

10. If $C_k = \binom{n}{k}$ for $n > 2$, $1 \leq k \leq n$, show that

$$\sum_{k=1}^{n} \sqrt{C_k} \leq \sqrt{n(2^n - 1)}$$