MA 491 problem set for November 28, 2006

More Inequalities

1. For positive numbers a and b, $a \neq b$, prove that $(ab^n)^{1/(n+1)} < \dfrac{a+nb}{n+1}$.

2. Prove that if $a_1, \ldots, a_n$ are real numbers with $a_1 + \ldots a_n = 1$, then $a_1^2 + \ldots + a_n^2 \geq 1/n$.

3. Prove that for $0 < x < \pi/2$, $\cos^2 x + x \sin x < 2$.

4. For each positive integer n, show that $\left(1+\dfrac{1}{n}\right)^n < \left(1+\dfrac{1}{n+1}\right)^{n+1}$.

5. Suppose that $f : R \to R$ satisfies

$$f\left(\frac{x+y}{2}\right) < \frac{f(x)+f(y)}{2}$$

for all $x,\ y \in (a,b),\ x \neq y$. Show that

$$f\left(\frac{x_1+x_2+\ldots+x_n}{n}\right) < \frac{f(x_1)+\ldots+f(x_n)}{n}$$

whenever the x_i's are in (a,b) with $x_i \neq x_j$ for at least one pair (i,j).

6. Putnam 1996 B2. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \ldots \cdot (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

7. Putnam 1988 B2. Prove or disprove: if x and y are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$.

8. Putnam 1999 B4. Can a countably infinite set have an uncountable collection of nonempty subsets such that the intersection of any two of them is finite? (This is not an inequality problem.)

9. Putnam 1990 A2. Is $\sqrt{2}$ the limit of a sequence of numbers of the form $\sqrt[3]{n} - \sqrt[3]{m}$, $(n,\ m = 0, 1, 2, \ldots)$?

10. Putnam 1992 A4. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1},\quad n = 1, 2, 3, \ldots$$

compute the values of the derivatives $f^{(k)}(0),\ k = 1, 2, 3, \ldots$.

SPIDERS! I may have messed up the spider problem. The classic problem has a room 12 feet high by 12 feet wide by 30 feet long with a fly one foot off the floor in the middle of the 12 foot side and a spider one foot from the ceiling in the middle of the opposite side. The question is then to find the shortest "spider path" from the spider to the fly.