

**640:491. Undergraduate Mathematics Problem Solving Seminar – Fall 2008**  
Problem Set 1 (Version 9/1/08; to be discussed beginning 9/2/08)

1. Find the smallest positive integers having exactly 100 divisors.
2. If  $S$  is a set of real numbers let  $S + S$  be the set of all sums of the form  $a + b$  where  $a \in S$  and  $b \in S$  (where  $a, b$  are allowed to be the same number). For each positive integer  $n$ , how should you choose  $S$  of size  $n$  if you want to minimize the size of  $S + S$ ?
3. Among all lists of positive integers that sum to 1000, what is the most their product can be?
4. Given 6 points in the plane, show that the ratio of the maximum distance between two of them to the minimum distance between two of them is at least  $\sqrt{3}$ .
5. Standard U.S. coins are pennies (1 cent), nickels (5 cents), dimes (10 cents), quarters (25 cents), half dollars (50 cents) and dollars (100 cents). Say that an integer  $k$  is *convertible* provided that it is possible to find a collection of  $k$  coins that adds up to a dollar. What is the smallest positive integer that is not convertible?
6. Consider the following lottery with 100 players. 100 slips of paper with the numbers 1 through 100 are placed in a hat and each player gets a random slip which he does not show to the other players. Each player has an option to keep his slip or to return it to the hat. (He must decide whether to do this without knowing what the other players are doing). Once each player decides, all players who return their slips to the hat choose again. The game continues until no player or only one player returns his slip to the hat. At that point each player gets paid the number of dollars corresponding to the number on his slip of paper.

Assuming all other players play their optimal strategy, what is the best strategy?

7. Determine all polynomials  $P(x)$  that satisfy the equations  $P(x^2 + 1) = (P(x))^2 + 1$  and  $P(0) = 0$ .
8. Let  $f$  and  $g$  be differentiable functions that satisfy  $f'(x)g(x) \neq f(x)g'(x)$  for all  $x$ . Prove that between any two roots of  $f$  there is a root of  $g$ .
9. Evaluate the determinant of the  $n \times n$  matrix whose  $(i, j)$ th entry is  $a^{|i-j|}$ .

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Assuming that the other people are typical of the population what strategy would you use? (This is not a mathematical question.)

10. Consider the following process on the interval  $[0, 1]$ . You select an infinite sequence  $P_0, P_1, P_2, \dots$  of distinct points from the interval  $[0, 1]$  where  $P_0 = 0$  and  $P_1 = 1$ . Each time you place a point you are paid a reward which is calculated as follows: After having selected the first  $P_0, P_1, \dots, P_i$  you have divided  $[0, 1]$  into  $i$  intervals. When you place  $P_{i+1}$  you divide one of the existing intervals into two intervals and your reward is the product  $ab(a + b)$  where  $a$  and  $b$  are the lengths of the two new intervals that you created. (i) What is the maximum total reward that you can obtain? (ii) Show that you get the maximum reward if you choose  $P_2, P_3, \dots$  to be any sequence that is *dense* in the interval  $[0, 1]$  (which means that for all  $x, y$  with  $0 < x < y < 1$  your sequence contains at least one point in the interval  $(x, y)$ ).

11. Is the infinite series:

$$\sum_{i=1}^{\infty} 1/n^{(n+1)/n}$$

convergent? Why or why not?

12. For which positive numbers  $s$  is it true that:

$$\lim_{n \rightarrow \infty} n^s \prod_{j=5}^n [1 - \frac{9}{2j}] = \infty.$$

13. Is it possible to find integers  $a, b, c$  each having absolute value less than 1,000,000 such that:

$$|a + b\sqrt{2} + c\sqrt{3}| < 10^{-11}?$$

14. For an integer  $n$ , let  $d(n)$  be the largest odd divisor of  $n$ , and let  $D(n) = \sum_{i=1}^n d(i)$ . Let  $T(n) = \sum_{i=1}^n i$ . Prove that there are infinitely many positive integers  $n$  such that  $3D(n) = 2T(n)$ .

15. Let us say that a positive real number  $x$  is a *Fermat number* if there are three distinct positive integers  $a, b, c$  such that  $a^x + b^x = c^x$ . Prove that there exist arbitrarily large Fermat numbers (which means that for every real number  $M$  there is a Fermat number  $x$  that is bigger than  $M$ ).

16. Suppose you want to cover a square of side one (the boundary and the interior) by three circles, not necessarily of the same area. How should the circles be chosen to minimize the sum of the areas of the circles.

17. Is there an integer  $n$  such that if you write  $2^n$  in base 10 the resulting sequence of digits includes 1000 consecutive 0's?