

## 501 Problems

**55-62.** Wheeden and Zygmund, pages 85-86. Problems 3, 6, 8, 12, 13, 14, 16, 18. (Note: 16 can be done easily by using the theorem proved in class: if  $f : S \rightarrow \mathbb{R}$  is a measurable function on the measure space  $(S, \Sigma, \mu)$  and  $M_f$  is the measure:

$$M_f(A) \triangleq \mu(\{s \in A; f(s) \in A\}), \quad A \subseteq \mathbb{R} \text{ is a Borel set,}$$

then 
$$\int_S f d\mu = \int_{\mathbb{R}} \alpha dM_f(\alpha) \quad (\text{if the integrals are defined.})$$

**63.** (Folland)(a) Modify the proof of the Dominated Convergence Theorem to prove: If  $\{f_n\}$  and  $\{g_n\}$  are sequences of integrable functions, with  $f_n \rightarrow f$  and  $g_n \rightarrow g$  a.e., if  $\int g d\mu < \infty$ , and if  $|f_n| \leq g_n$  for every  $n$ , then  $\int f_n d\mu \rightarrow \int f d\mu$ .

(b) Suppose  $\{f_n\}$  and  $f$  are all integrable and  $f_n \rightarrow f$  a.e., prove that

$$\int |f - f_n| d\mu \rightarrow 0 \quad \text{if and only if} \quad \int |f_n| d\mu \rightarrow \int |f| d\mu.$$

**64.** (Folland) Show that if  $\int_{\mathbb{R}} |f| dm < \infty$ , then the function  $x \rightarrow F(x) \triangleq \int_{(-\infty, x]} f dm$  is continuous.

**65.** (Folland) Compute with justifications the limit

$$\lim_{n \rightarrow \infty} \int_a^\infty \frac{n}{1 + n^2 x^2} dx,$$

in the cases  $a > 0$ ,  $a = 0$ , and  $a < 0$ .

**66.** (Folland) Show with justification that

$$\frac{d}{dt} \int_0^\infty e^{-tx} dx = \int_0^\infty (-x)^n e^{-tx} dx.$$

Use this to show the formula:  $\int_0^\infty x^n e^{-x} dx = n!$ . Now prove, with justification of the limit operations,

$$\lim_{k \rightarrow \infty} \int_0^k x^n \left(1 - \frac{x}{k}\right)^k dx = n!.$$