

501 Measures and construction of measures

1 Measures, finitely additive measures, and the extension problem

In this lecture S will be a fixed, non-empty set, \mathcal{R} will denote an algebra of S , and Σ a σ -algebra of S .

If $\{A_n\}$ is a sequence of sets, the notation $A_n \uparrow A$ will indicate that the sequence is increasing, that is, $A_1 \subseteq A_2 \subseteq \dots$, and that $A = \bigcup_1^\infty A_n$. The notation $A_n \downarrow A$ will indicate that the sequence is decreasing, that is, $A_1 \supseteq A_2 \supseteq \dots$, and that $A = \bigcap_1^\infty A_n$.

The following constructions will be used several times. Suppose $\{A_n\}$ is a sequence of sets in an algebra or σ -algebra, and $A_n \uparrow A$. If $B_1 \triangleq A_1$, and for $n > 1$, $B_n \triangleq A_n - A_{n-1}$, then:

1. B_1, B_2, \dots are disjoint sets in the algebra or σ -algebra, and $B_n \subset A_n$ for each $n \geq 1$;
2. $A_n = \bigcup_1^n B_i$, for each $n \geq 1$;
3. $A = \bigcup_1^\infty B_n$.

Conversely, if B_1, B_2, \dots are disjoint sets and $A_n \triangleq \bigcup_1^n B_i$, then $A_n \uparrow \bigcup_1^\infty B_n$. This observation leads easily to the following.

Lemma 1 *An algebra which is closed under countable disjoint unions is a σ -algebra.*

Recall that a finitely additive measure $\overset{\circ}{\mu}$ on (S, \mathcal{R}) satisfies (i) $\overset{\circ}{\mu}(\emptyset) = 0$; (ii) $\overset{\circ}{\mu}(B) \in [0, \infty]$ for every $A \in \mathcal{R}$; and, (iii) (finite additivity) $\overset{\circ}{\mu}(\cup_1^n B_i) = \sum_1^n \overset{\circ}{\mu}(B_i)$, if B_1, \dots, B_n is a disjoint sequence of sets in \mathcal{R} .

A measure μ on (S, Σ) satisfies (i) $\mu(\emptyset) = 0$; $\mu(B) \in [0, \infty]$ for every $A \in \Sigma$; and, (iii) (countable additivity) $\mu(\cup_1^\infty B_i) = \sum_1^\infty \mu(B_i)$, if B_1, B_2, \dots is a countable disjoint sequence of sets in Σ .

The finite additivity of a measure or finitely additive measure implies the following monotonicity property; (this is stated for a measure only, but is true for a finitely additive measure if the σ -algebra Σ is replaced by the algebra \mathcal{R}):

$$\text{if } A, B \in \Sigma \text{ and } A \subseteq B, \text{ then } \mu(A) \leq \mu(B). \quad (1)$$

This is an easy consequence of

$$\mu(B) = \mu(A) + \mu(B - A),$$

which follows from additivity, and of the positivity of μ .

Lemma 2 *A finitely additive measure $\overset{\circ}{\mu}$ is finitely subadditive, that is, if A_1, \dots, A_n are sets in \mathcal{R} ,*

$$\overset{\circ}{\mu} \left(\bigcup_1^n A_i \right) \leq \sum_1^n \overset{\circ}{\mu} (A_i) \quad (2)$$

Likewise, a measure μ on (S, Σ) is countably subadditive: if A_1, A_2, \dots is a sequence of sets in Σ ,

$$\mu \left(\bigcup_1^\infty A_i \right) \leq \sum_1^\infty \mu(A_i) \quad (3)$$

Proof. Let $B_1 = A_1$, and for $i \geq 1$, let $B_i = \bigcup_{j=1}^i A_j - \bigcup_{j=1}^{i-1} A_j$. Noting that B_1, \dots, B_n are disjoint, that $B_i \subseteq A_i$ for each $1 \leq i \leq n$, and that $\bigcup_1^n B_i = \bigcup_1^n A_i$,

$$\overset{\circ}{\mu} \left(\bigcup_1^n A_i \right) = \overset{\circ}{\mu} \left(\bigcup_1^n B_i \right) = \sum_1^n \overset{\circ}{\mu} (B_i) \leq \sum_1^n \overset{\circ}{\mu} (A_i)$$

The last step uses (1) applied to $\overset{\circ}{\mu}$. The proof of the countable subadditivity property for measures is similar. \diamond

Example: Step 1 in the construction of Lebesgue measure

Let \mathcal{V}^d be the algebra of subsets of \mathbb{R}^d consisting of all finite disjoint unions of left half-open rectangle—that is rectangles of the form

$$V = (r_1, s_1] \times \cdots \times (r_d, s_d] \cap \mathbb{R}^d,$$

where $-\infty \leq r_i \leq s_i \leq \infty$ for each i . For such a V , define $\overset{\circ}{m}(V)$ to be the volume of V ; $\overset{\circ}{m}(V) \triangleq \prod_1^d (s_i - r_i)$. If $\bigcup_1^n V_i$ is a finite, disjoint union of such rectangles, define $\overset{\circ}{m} \left(\bigcup_1^n V_i \right) = \sum_1^n \overset{\circ}{m}(V_i)$. We claim that $\overset{\circ}{m}$ is a well-defined, finitely additive measure on \mathcal{V}^d .

To see that $\overset{\circ}{m}$ is well-defined we must show that if V_1, \dots, V_n are disjoint, left-half open rectangles, and so also are W_1, \dots, W_m , then

$$\sum_1^n \overset{\circ}{m}(V_i) = \sum_1^m \overset{\circ}{m}(W_j).$$

Since the collection of left-half open rectangle is closed under finite intersections, the formula,

$$V_i = \bigcup_1^m V_i \cap W_j$$

represents each rectangle V_i as a finite disjoint union of a finite number of left-half open intervals, and for such a decomposition, the equality

$$\overset{\circ}{m}(V_i) = \sum_{j=1}^m \overset{\circ}{m}(V_i \cap W_j)$$

is valid. Thus

$$\sum_1^n \overset{\circ}{m}(V_i) = \sum_{i=1}^n \sum_{j=1}^m \overset{\circ}{m}(V_i \cap W_j).$$

But the same reasoning, with the roles of V and W interchanged gives,

$$\sum_1^m \overset{\circ}{m}(W_j) = \sum_{j=1}^m \sum_{i=1}^n \overset{\circ}{m}(V_i \cap W_j),$$

which is the same.

The finite additivity of $\overset{\circ}{m}$ is a consequence of the finite additivity built into its definition; the proof is left as an exercise.

Example 2. Coin-tossing measure for a fair coin. We pick up on example 5 of section 2 of the second set of lecture notes. There $\Omega \triangleq \{0, 1\}^\infty$ is a model for the set of possible outcomes of an infinite sequence of tosses of a coin. We defined the algebra $\mathcal{A} \triangleq \bigcup_{n \geq 1} \mathcal{A}_n$, where for each n , \mathcal{A}_n is the σ -algebra $\pi_n^{-1}(\mathcal{P}(\{0, 1\}^n))$, (where π_n is projection onto the first n coordinates). In other words. the algebra \mathcal{A} is the collection of subsets of Ω of the form $\{\omega; (\omega_1, \dots, \omega_n) \in B\}$, as n ranges over the positive integers, and, for each n , B ranges over the subsets of $\{0, 1\}^n$.

Given a positive integer n , and a set $B \subseteq \{0, 1\}^n$, what is the probability that the first n tosses of a fair coin yields a sequence falling in B ? Since each sequence of n tosses should be equally likely and there are 2^n possible sequences, this probability should be $|B|/2^n$, where $|B|$ denotes the cardinality of B . This leads to the following definition of a finitely additive probability measure $\overset{\circ}{P}$ on \mathcal{A} :

$$\text{for any } n \geq 1 \text{ and any } B \subseteq \{0, 1\}^n, \quad \overset{\circ}{P}(\pi_n^{-1}(B)) \triangleq \frac{|B|}{2^n}.$$

Of course, it needs to be checked that $\overset{\circ}{P}$ is consistently defined and is finitely additive.

To check consistency, suppose that $\pi_n^{-1}(B_1) = \pi_m^{-1}(B_2)$, where $B_1 \subseteq \{0, 1\}^n$ and $B_2 \subseteq \{0, 1\}^m$. If $n = m$ one may check that then, necessarily $B_1 = B_2$; if, without

loss of generality, $m > n$, one may check that $B_2 = B_1 \times \{0, 1\}^{m-n}$. Thus, in either case

$$\frac{|B_2|}{2^m} = \frac{|B_1|2^{m-n}}{2^m} = \frac{|B_1|}{2^n},$$

and the definition of $\overset{\circ}{P}$ is consistent.

To prove the finite additivity, consider a finite disjoint collection A_1, \dots, A_m of sets in \mathcal{A} . There is a positive integer N so that all these sets are in \mathcal{A}_N . Thus there are subsets B_1, \dots, B_m of $\{0, 1\}^N$ such that $A_i = \pi_N^{-1}(B_i)$ for $1 \leq i \leq m$. Moreover, B_1, \dots, B_m must be disjoint, and $\bigcup_1^m A_i = \pi_N^{-1}(\bigcup_1^m B_i)$. Thus

$$\overset{\circ}{P}\left(\bigcup_1^m A_i\right) = \frac{|\bigcup_1^m B_i|}{2^N} = \sum_1^m \frac{|B_i|}{2^N} = \sum_1^m \overset{\circ}{P}(A_i).$$

The extension problem. Given a finitely additive measure $\overset{\circ}{\mu}$ on an algebra \mathcal{R} , under what conditions does there exist a measure μ defined on $\sigma(\mathcal{R})$ that extends $\overset{\circ}{\mu}$ in the sense that

$$\mu(A) = \overset{\circ}{\mu}(A) \quad \text{for every } A \in \mathcal{R}?$$

If such an extension exists, is it unique and how can the extension be constructed? To begin to answer this question, it is necessary first to understand the elementary properties of measures.

2 Elementary general properties of measures.

Theorem 1 *Let μ be a measure on (S, Σ) .*

(i) μ is monotone and subadditive.

(ii) (Continuity from below) *If $\{A_n\}_{n \geq 1} \subset \Sigma$ and $A_n \uparrow A$, then $\mu(A_n) \uparrow \mu(A)$.*

(iii) (Continuity from above) *If $\{A_n\}_{n \geq 1} \subset \Sigma$ and if $\mu(A_m) < \infty$ for at least one m , $A_n \downarrow A$, then $\mu(A_n) \downarrow \mu(A)$.*

Proof: Monotonicity and subadditivity were treated in the previous section.

To argue continuity from below, given the increasing sequence of sets $\{A_n\}$, define the disjoint sequence of sets $\{B_n\}$, where $B_1 = A_1$ and $B_n = A_n - A_{n-1}$ for $n > 1$. Then, using countable additivity of μ ,

$$\mu(A) = \mu\left(\bigcup_1^\infty B_k\right) = \sum_1^\infty \mu(B_k) = \lim_{n \rightarrow \infty} \sum_1^n \mu(B_k) = \lim_{n \rightarrow \infty} \mu\left(\bigcup_1^n B_k\right) = \lim_{n \rightarrow \infty} \mu(A_n).$$

For continuity from above, assume without loss of generality that $\mu(A_1) < \infty$. If the sequence $\{A_n\}$ is decreasing to A , then $\{A_1 - A_n\}$ increases to $A_1 - A$, and so continuity from above follows by applying continuity from below to $\{A_1 - A_n\}$. \diamond

From the proof of the theorem, one sees that the monotone continuity properties of a measure are a direct consequence of countable additivity. In fact monotone continuity and countable additivity properties are equivalent in the sense made precise in the following result. A measure, or finitely additive measure, for which $\mu(S) < \infty$ is called finite.

Lemma 3 *Let μ be a finitely additive measure on a σ -algebra Σ . The following are equivalent.*

(1) μ is countably additive.

(2) μ is continuous from below.

If the further assumption that μ is finite is made, (1) and (2) are equivalent to

(3) (Continuity from above at \emptyset) If $\{A_n\}_{n \geq 1}$ is a sequence of sets in Σ and if $A_n \downarrow \emptyset$, then $\mu(A_n) \downarrow 0$.

Proof: The previous theorem shows that (1) implies (2). To show that (2) implies (1), observe that if $\{B_n\}_{n \geq 1}$ is a disjoint sequence of sets in Σ , and if $A_n = \cup_1^n B_i$, then $A_n \uparrow \cup_1^\infty B_i$. Thus, assuming continuity from below and using the assumed finite additivity of μ ,

$$\mu\left(\bigcup_1^\infty B_i\right) = \lim_{n \rightarrow \infty} \mu(A_n) = \lim_{n \rightarrow \infty} \sum_1^n \mu(B_i) = \sup_1^\infty \mu(B_i),$$

thereby proving countable additivity.

Now assume that μ is finitely additive and that μ is finite, i.e., $\mu(S) < \infty$. Suppose that $A_n \uparrow A$. Then $A - A_n \downarrow \emptyset$. Since $\mu(A)$ is perforce finite, $\mu(A) - \mu(A_n) = \mu(A - A_n)$ for all n . Thus if (3) holds, $\lim_{n \rightarrow \infty} \mu(A) - \mu(A_n) = 0$, proving continuity from below. \diamond

Remark. From the proof of Lemma 3, If μ is finitely additive on (S, Σ) and (3) is true, if $A_n \uparrow A$ and $\mu(A) < \infty$, where $\{A_n\}_{n \geq 1}$ is a sequence in Σ , then $\mu(A_n) \uparrow \mu(A)$.

3 Continuity from below for algebras

Let $\overset{\circ}{\mu}$ be a finitely additive measure on (S, \mathcal{R}) where \mathcal{R} is an algebra. It is natural to make the following definitions.

Definition. $\overset{\circ}{\mu}$ is said to be continuous from below if, given a sequence of sets $\{A_n\}_{n \geq 1}$, each of which is in \mathcal{R} , and such that $A_n \uparrow A$ and $A \in \mathcal{R}$, $\overset{\circ}{\mu}(A_n) \uparrow \overset{\circ}{\mu}(A)$.

$\overset{\circ}{\mu}$ is said to be continuous from above at \emptyset if, given a sequence of sets $\{A_n\}_{n \geq 1}$, each of which is in \mathcal{R} , and such that $A_n \downarrow \emptyset$, then $\overset{\circ}{\mu}(A_n) \downarrow 0$.

Here is the point of this definition. Suppose $\overset{\circ}{\mu}$ admits an extension to a measure μ on $(S, \sigma(\mathcal{R}))$. Then Lemma 3 clearly implies that $\overset{\circ}{\mu}$ must be continuous from below. Thus continuity from below is a necessary condition for existence of an extension. We will see that it is also a sufficient condition. Hence to finish constructing measures, for example in Examples 1 and 2, it will only be necessary to check continuity from below of the finitely additive measures.

The following lemma is proved exactly as Lemma 3

Lemma 4 *For a finitely additive measure $\overset{\circ}{\mu}$ on (S, \mathcal{R}) , the following are equivalent:*

(a) $\overset{\circ}{\mu}$ is continuous from below.

(b) (Countable additivity in \mathcal{R}). Given a disjoint sequence $\{B_n\}_{n \geq 1}$ of sets in \mathcal{R} such that $\cup_1^\infty B_i$ is also in \mathcal{R} , $\overset{\circ}{\mu}(\cup_1^\infty B_n) = \sum_1^\infty \overset{\circ}{\mu}(B_n)$.

If $\overset{\circ}{\mu}$ is continuous from above at \emptyset and if $A_n \uparrow A$ where each A_n , $n \geq 1$ and also A are in \mathcal{R} , and where $\overset{\circ}{\mu}(A) < \infty$, then $\overset{\circ}{\mu}(A_n) \uparrow \overset{\circ}{\mu}(A)$. \diamond

Example 1 (continued)

Proposition 1 *The finitely additive measure $\overset{\circ}{m}$ on $(\mathbb{R}^d, \mathcal{V}^d)$, as constructed in example 1 based on the volume of rectangles is continuous from below.*

This result will be proved later. For now, we note that to prove the proposition, it suffices to prove continuity from below in the case that the limit set A has finite $\overset{\circ}{m}$ measure, and therefore, by the remark to Lemma 3, to prove continuity from above at \emptyset . To see this, suppose $A_n \uparrow A$, where all sets are in \mathcal{V}^d . The set A is a finite disjoint union $A = \cup_1^r V_i$ of left-half open rectangles. Consider rectangle V_i . Then for each n , every one of the rectangles making up A_n either falls entirely in V_i or is disjoint with it, and $\overset{\circ}{m}(A_n) = \sum_1^r \overset{\circ}{m}(A_n \cap V_i)$. Hence to prove the continuity from below it suffices to prove that $\overset{\circ}{m}(A_n \cap V_i) \uparrow \overset{\circ}{m}(V_i)$ for each i . In other words, it suffices to prove continuity from below for the case when A is a rectangle in \mathcal{V}^d . We will show that if continuity from below is true for rectangles of finite $\overset{\circ}{m}$ measure then it is true also for rectangles of infinite $\overset{\circ}{m}$ measure. We can choose an increasing sequence of rectangles B_k such that $B_k \uparrow A$, and $\overset{\circ}{m}(B_k) < \infty$ for every k . Since at least one of the sides of A has infinite length, the corresponding sides of the B_k sequence must increase to an interval of infinite length and therefore $\overset{\circ}{m}(B_k) \uparrow \infty$. Now let $\{A_n\}$ be any sequence of sets in \mathcal{V}^d with $A_n \uparrow A$. Since $\overset{\circ}{m}(B_k) < \infty$ for every k , it follows by assumption that $\lim_{n \rightarrow \infty} \overset{\circ}{m}(A_n \cap B_k) = \overset{\circ}{m}(B_k)$. Thus, for each k , $\lim_{n \rightarrow \infty} \overset{\circ}{m}(A_n) \geq \lim_{n \rightarrow \infty} \overset{\circ}{m}(A_n \cap B_k) = \overset{\circ}{m}(B_k)$. Letting $k \uparrow \infty$, implies $\lim_{n \rightarrow \infty} \overset{\circ}{m}(A_n) = \infty$, which is what we needed to prove. \diamond

Example 2(continued)

Proposition 2 Any finite, finitely additive measure on (Ω, \mathcal{A}) is continuous from below.

This will be proved later. Again, since continuity from below is equivalent to continuity from above at \emptyset for finite, finitely additive measures, it will suffice to prove continuity from above at \emptyset .

As a consequence of Proposition 2, the finitely additive coin tossing measure $\overset{\circ}{P}$, which is finite because $\overset{\circ}{P}(\Omega) = 1$, is continuous from below on $(\Omega, \mathcal{A}_\infty)$. \diamond

4 The extension theorem.

The next theorem states when an extension of $\overset{\circ}{\mu}$ to a measure exists and describes how to construct it. It is one of the big theorems of abstract measure theory and is due mostly to Carathéodory, building on the ideas of Lebesgue.

To speak of extensions more generally, we shall say that a measure space (S, Σ, μ) is an extension of $(S, \mathcal{R}, \overset{\circ}{\mu})$ if $\mathcal{R} \subset \Sigma$ and μ and $\overset{\circ}{\mu}$ agree on \mathcal{R} . (This definition does not exclude the case in which \mathcal{R} is a σ -algebra and $\overset{\circ}{\mu}$ is actually a measure, and we may have occasion to use it in this sense. But for this discussion, we are interested in the case when \mathcal{R} is only an algebra.)

The extension theorem requires one new definition: a measure μ on (S, Σ) is called σ -finite if there exists a countable sequence of sets $\{E_n\}$ all in Σ , such that $S = \bigcap_1^\infty E_n$ and $\mu(E_n) < \infty$ for each n . Using the same definition, one can likewise apply the term σ -finite to a finitely additive measure. The finitely additive measure $\overset{\circ}{m}$ of example 1 is σ -finite, as are most measures encountered in practice.

The extension theorem requires prescribing how, starting from $\overset{\circ}{\mu}$, one defines the measure of a set not in \mathcal{R} . This is accomplished by the *outer measure* induced by $\overset{\circ}{\mu}$. Later, a general definition and theory of outer measure will be developed. For now, only the outer measure induced by $\overset{\circ}{\mu}$ is defined. The idea is to use a kind of outer sum approximation.

Definition. The outer measure induced by μ on (S, \mathcal{R}) is the function μ^* which assigns to every non-empty subset $A \subseteq S$, the quantity

$$\mu^*(A) = \inf \left\{ \sum_1^\infty \overset{\circ}{\mu}(E_i); A \subseteq \bigcup_1^\infty E_i, E_i \in \mathcal{R} \forall i \right\}.$$

Also $\mu^*(\emptyset) \triangleq 0$.

It is necessary also to single out a class of subsets of S that have a strong finite additivity property with respect to the outer measure.

Definition. A set A is μ^* -measurable if

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for every } E \subset S.$$

Let \mathcal{M} denote the collection of all μ^* -measurable sets

In general, not all subsets of S will be μ^* -measurable.

Now we can state the extension theorem.

Theorem 2 Let $\overset{\circ}{\mu}$ be a finitely additive measure on (S, \mathcal{R}) , where \mathcal{R} is an algebra.

(a) The collection \mathcal{M} of μ^* -measurable sets form a σ -algebra that contains \mathcal{R} , and therefore that contains $\sigma(\mathcal{R})$.

(b) The outer measure μ^* is a measure on (S, \mathcal{M}) ; that is μ^* is countably additive when restricted to \mathcal{M} . By (a), μ^* is then also a measure on $(S, \sigma(\mathcal{R}))$

(c) If $\overset{\circ}{\mu}$ is continuous from below, then (S, \mathcal{M}, μ^*) and $(S, \sigma(\mathcal{R}), \mu^*)$ are extensions of $(S, \mathcal{R}, \overset{\circ}{\mu})$.

(d) If $\overset{\circ}{\mu}$ is continuous from below and σ -finite, (S, \mathcal{M}, μ^*) is its unique extension to (S, \mathcal{M}) .

To summarize results (c) and (d) of the theorem, if $\overset{\circ}{\mu}$ is continuous from below and σ -finite, it admits a unique extension to a measure on $(S, \sigma(\mathcal{R}))$ and that extension is provided by the outer measure. The theorem actually provides an extension on the potentially larger σ -algebra of μ^* -measurable sets. In this discussion, we shall use μ to denote the outer measure when restricted to \mathcal{M} (or \mathcal{R}); this will help differentiate between the outer measure μ^* , which is defined on all subsets of S , and its restriction to a σ -algebra on which it is a measure.

Example 1 (continued). *Lebesgue outer measure* is the outer measure induced by $\overset{\circ}{m}$: for any $A \subset \mathbb{R}^d$,

$$m^*(A) \triangleq \inf \left\{ \sum_1^d \overset{\circ}{m}(E_i); A \subseteq \bigcup_1^\infty E_i, E_i \in \mathcal{V}^d \forall i \right\}.$$

The m^* measurable sets are called the Lebesgue measurable sets. For convenience of notation, let $\mathcal{L}(\mathbb{R}^d)$ denote the Lebesgue measurable sets of \mathbb{R}^d .

Theorem 2 (a) implies that the collection Borel sets of \mathbb{R}^d , $\mathcal{B}(\mathbb{R}^d)$, which is generated by \mathcal{V}^d , by is contained in $\mathcal{L}(\mathbb{R}^d)$, the Lebesgue measurable sets; part (b),(c), and (d), together with with Proposition 1, imply that $(\mathbb{R}^d, \mathcal{L}(\mathbb{R}^d), m^*)$ is the unique extension of $\overset{\circ}{m}$. We shall use m to denote m^* restricted to $\mathcal{L}(\mathbb{R}^d)$, in order to distinguish between the outer measure as defined as a function on all subsets of \mathbb{R}^d , and its restriction to Lebesgue measurable sets. The measure m is called the Lebesgue measure. For almost all purposes of analysis, one can work with the measure space

$(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), m)$, in which the σ -algebra is restricted to the Borel sets and sometimes. We still refer m as Lebesgue measure even when considering its restriction to the Borel sets; it should be clear from context whether Lebesgue measurable or Borel sets are the base σ -algebra.

It will turn out that there is a close relationship between $\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{L}(\mathbb{R}^d)$ in terms of m ; for every Lebesgue measurable set A , there is a Borel set B such that A and B differ by a set of measure zero. \diamond

Example 2 (continued). From Proposition 2 and Theorem 2, it follows that the finitely additive measure $\overset{\circ}{P}$ admits a unique extension P , a measure on $(\Omega, \mathcal{A}_\infty)$; for $A \in \mathcal{A}_\infty$,

$$P(A) = \inf \left\{ \sum_1^\infty \overset{\circ}{P}(E_i); A \subseteq \bigcap_1^\infty E_i, E_i \in \mathcal{A} \forall i \right\}.$$

Usually, one is content to work with $(\Omega, \mathcal{A}, \infty P)$, and does not bother with the collection of all P^* -measurable sets. \diamond

5 Proofs of Propositions 1 and 2

5.1 Proof of Proposition 1.

We shall treat the case $d = 1$. The proof of the case $d > 1$ is only a matter of more complicated notation! From the discussion in section 3, it suffices to show that $\overset{\circ}{m}$ is continuous from above at \emptyset . Let $\{A_n\}_{n \geq 1}$ be a decreasing sequence of sets in \mathcal{V}^1 with $\overset{\circ}{m}(A_1) < \infty$ and suppose $A_n \downarrow \emptyset$. Let $\epsilon > 0$.

The essential point is this: for each n there is a set F_n in \mathcal{V}^1 such that its closure \bar{F}_n is a subset of A_n and also $\overset{\circ}{m}(A_n - F_n) < \epsilon/2^n$. (Note that $A_n - F_n \in \mathcal{V}^1$ since \mathcal{V}^1 is an algebra.) This is easy to see. Each A_n is a disjoint union of the form, $A_n = \bigcup_1^m (a_i, b_i]$, where each interval $(a_i, b_i]$ is finite. Let $F_n \triangleq \bigcup_1^m (a_i + \delta_i, b_i]$, where $0 < \delta_i < 1/(m2^n)$, $1 \leq i \leq m$.

Next, let $K_n \triangleq \bigcap_1^n F_i$. Again $\bar{K}_n \subset A_n$ for each n . Also, using finite subadditivity of $\overset{\circ}{m}$,

$$\overset{\circ}{m}(A_n - K_n) = \overset{\circ}{m} \left(\bigcup_1^n A_n - F_i \right) \leq \overset{\circ}{m} \left(\bigcup_1^n A_i - F_i \right) \leq \sum_1^n \overset{\circ}{m}(A_i - F_i).$$

Thus $\overset{\circ}{m}(A_n - K_n) < \sum_1^n \frac{\epsilon}{2^i} < \epsilon$.

Now $\{\bar{K}_n\}_{n \geq 1}$ is a decreasing sequence of compact sets with $\bigcap_1^\infty \bar{K}_n \subset \bigcap_1^\infty A_n = \emptyset$. By the finite intersection property of compact sets, there exists N such that, $K_N = \emptyset$,

and hence $\overset{\circ}{m}(K_N) = 0$. Therefore, for $n \geq N$,

$$\overset{\circ}{m}(A_n) \leq \overset{\circ}{m}(A_N) = \overset{\circ}{m}(A_N - K_N) + \overset{\circ}{m}(K_N) < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, $\overset{\circ}{m}(A_n) \downarrow 0$. Therefore, $\overset{\circ}{m}$ is continuous from above at \emptyset .

5.2 Ian Levitt's proof of Proposition 1 ($d = 1$)

We prove directly that $\overset{\circ}{m}$ is continuous from below. By the remarks in section 3, it suffices to treat the case in which $A_n \uparrow A$ and A is a finite interval $A = (a, b]$. By Lemma 4 b), it suffices to prove that if

$$(a, b] = \bigcap_1^{\infty} I_n,$$

where $\{I_n\}_{n \geq 1}$ is a disjoint sequence of sets in \mathcal{V}^1 , then

$$b - a = \sum_1^{\infty} \overset{\circ}{m}(I_n).$$

It actually suffices to consider the case when each I_n is an interval $(a_n, b_n]$, since sets in \mathcal{V}^1 are finite disjoint unions of such intervals. Let

$$G \triangleq \left\{ x; a \leq x \leq b, b - y = \sum_1^{\infty} \overset{\circ}{m}((y, b] \cap I_n) \quad \forall y, x \leq y \leq b \right\}.$$

In words, x is in G if countable additivity for the decomposition $(y, b] = \cup_1^{\infty} (y, b] \cap I_n$ holds for $\overset{\circ}{m}$ for all $x \leq y \leq b$. We will show that G is both open and closed with respect to the relative topology on $[a, b]$. Since $[a, b]$ is connected, this will prove that $G = [a, b]$.

Let $x \in G$. For some i , $a_i \leq x < b_i$. If $a_i < x < b_i$, then it is clear that a_i must also be in G , because for $a_i \leq y < x$,

$$b - y = b - x + x - y = \left[b_i - x + \sum_{j, a_j \geq x} (b_j - a_j) \right] + x - y = b_i - y + \sum_{j, (a_j \geq y)} (b_j - a_j).$$

If $x = a_i$, then there is a j so that $b_j = a_i$, and it follows similarly that $a_j < b_j = a_i$ is in G . Therefore G is open.

Let $\{x_n\}_{n \geq 1}$ be a sequence in G such that $x_n \downarrow x$. To show G is closed we must prove that $x \in G$. We may assume that $x_n > x$ for all n , for otherwise $x \in G$ automatically. If there is an i such that $a_i \leq x < b_i$, then for sufficiently large n , $x < x_n < b_i$, and by the previous argument, a_i and hence x are certainly in G . So suppose $x = b_i$ for some i . For each n ,

$$b - x_n = \sum_{i, a_i \geq x_n} (b_i - a_i) + \inf\{b_j; b_j \geq x_n\} - x_n.$$

As $n \rightarrow \infty$, this tends to

$$\sum_{i \in a_i > x} (b_i - a_i) + \inf\{b_j; b_j > x\} - x \quad (4)$$

In the case that $\bar{b} \triangleq \inf\{b_j; b_j > x\} > x$, $(x, \bar{b}]$ must in fact be one of the intervals $(a_k, b_k]$ and so (4) shows that $x \in G$. If $\bar{b} = x$, then $\bigcup_{i, a_i > x} (a_i, b_i] = (x, b]$, and again

(4) shows that $x \in G$.

5.3 Proof of Proposition 2

We will show that if $\{A_n\}_{n \geq 1}$ is a decreasing sequence of sets in the algebra \mathcal{A} , and if $A_n \downarrow \emptyset$, then in fact, there is an N so that $A_N = \emptyset$. It follows at once that any finitely additive measure on \mathcal{A} is continuous from above at \emptyset , and for finite measures, this suffices to prove that the finitely additive measure is continuous from below.

Consider $\{0, 1\}$ to be a topological space with the discrete topology— $\{0\}$ and $\{1\}$ are each open and closed at the same time. With this topology, $\{0, 1\}$ is compact. By Tychonoff's theorem, $\Omega = \{0, 1\}^\infty$ is compact in the product topology. Each set of the algebra \mathcal{A} is a closed, and so compact, set of the product topology on $\{0, 1\}$. Therefore if $A_n \downarrow \emptyset$, there is an N so that $A_N = \emptyset$.

The space $\{0, 1\}^\infty$ is simple enough that we could do a direct proof and avoid citing such a general result as Tychonoff's theorem. Here is a sketch; the student should fill in the details. If $A \subset \{0, 1\}^\infty$, and (w_1, \dots, w_n) is a string of 0's and 1's, let $A(w_1, \dots, w_n)$ denote the section at (w_1, \dots, w_n) ,

$$A(w_1, \dots, w_n) = \{(\omega_1, \omega_2, \dots); (w_1, \dots, w_n, \omega_1, \omega_2, \dots) \in A\}.$$

Notice that if $A \in \mathcal{A}_n$ then either $A(w_1, \dots, w_n) = \emptyset$ or $A(w_1, \dots, w_n) = \{0, 1\}^\infty$.

Assume that $A_i \downarrow A$, where each A_i is in \mathcal{A} . We will show that if A_i is non-empty for each i , then so is A . Define η_1, η_2, \dots recursively as follows. First $\eta_1 = 0$ if $A_i(0)$ is non-empty for all i , and $\eta_1 = 1$ if not. Note that when $\eta_1 = 1$, then in fact, $A_i(1)$ is non-empty for all i . To see this, observe that $A_i = [\{0\} \times A_i(0)] \cup [\{1\} \times A_i(1)]$ and use the assumption that A_i is non-empty for all i . Now define η_n recursively by setting $\eta_{n+1} = 0$ if $A_i(w_1, \dots, w_n, 0)$ is non-empty for all i , and otherwise let $\eta_{n+1} = 1$. Then $(\eta_1, \eta_2, \dots) \in \bigcap_1^\infty A_i$.

6 Proof of Theorem 2 and refinements

6.1 Outer measures

This section is preparatory to the proof of Theorem 2.

Definition. A function ν from the power set $\mathcal{P}(S)$ of a non-empty set S to $[0, \infty]$ is called an outer measure if

- (i) $\nu(\emptyset) = 0$.
- (ii) ν is monotone: $A \subset B$ implies $\nu(A) \leq \nu(B)$.
- (iii) ν is subadditive: if $\{A_n\}$ is a sequence of subsets of S , $\nu(\cup A_n) \leq \sum \nu(A_n)$.

Example. The outer measure μ^* induced by a finitely additive measure is an outer measure in the sense of this definition. The proof is left as an exercise.

Definition. Given an outer measure ν on S , a subset A of S is called ν -measurable if

$$\nu(E) = \nu(E \cap A) + \nu(E \cap A^c) \quad \text{for all subsets } E \text{ of } S.$$

Observe that since $\nu(E) \leq \nu(E \cap A) + \nu(E \cap A^c)$ automatically by subadditivity, to demonstrate ν -measurability of a set A , it is only necessary to show $\nu(E) \geq \nu(E \cap A) + \nu(E \cap A^c)$ for all subsets E of A .

Proposition 3 *The ν -measurable sets of an outer measure form a σ -algebra, on which ν is a countably additive measure.*

Proof. Let \mathcal{M} denote the collection of ν -measurable sets. The proof requires showing first that

$$A \in \mathcal{M} \text{ and } B \in \mathcal{M} \text{ imply } A \cup B \in \mathcal{M}. \quad (5)$$

Since \mathcal{M} is closed under complements by the symmetry in A and A^c of the definition of a ν -measurable set, (5) implies that

$$\mathcal{M} \text{ is an algebra} \quad (6)$$

Second, the proof requires showing that if A_1, \dots, A_n are disjoint sets in \mathcal{M} , then

$$\nu\left(E \cap \left[\bigcup_1^n A_i\right]\right) = \sum_1^n \nu(E \cap A_i) \quad (7)$$

To prove (5) write $E \cap [A \cup B] = (E \cap A \cap B) \cup (E \cap B \cap A^c) \cup (E \cap A \cap B^c)$. Use the assumption that both A and B are measurable and use subadditivity:

$$\begin{aligned} & \nu(E \cap [A \cup B]) + \nu(E \cap A^c \cap B^c) \\ & \leq \nu((E \cap A \cap B) + \nu(E \cap B \cap A^c) + \nu(E \cap A \cap B^c) + \nu(E \cap A^c \cap B^c) \\ & = \nu(E \cap B) + \nu(E \cap B^c) = \nu(E) \end{aligned}$$

To prove (7), observe that if $A \in \mathcal{M}$ and $B \cap A = \emptyset$,

$$\begin{aligned}\nu(E \cap [A \cup B]) &= \nu(E \cap [A \cup B] \cap A) + \nu(E \cap [A \cup B] \cap A^c) \\ &= \nu(E \cap A) + \nu(E \cap B)\end{aligned}$$

The general case of (7) is proved by induction on the number of sets.

To finish the proof, first note that (7) actually generalizes easily to a countable number of sets. If $\{A_n\}$ is a countable sequence of disjoint sets in \mathcal{M} , (7), monotonicity, and subadditivity imply

$$\sum_1^N \nu(E \cap A_n) = \nu\left(\bigcup_1^N (E \cap A_n)\right) \leq \sum_1^\infty \nu(E \cap A_n).$$

Taking $n \rightarrow \infty$ yields

$$\sum_1^\infty \nu(E \cap A_n) = \nu\left(\bigcup_1^\infty (E \cap A_n)\right). \quad (8)$$

Since we know already that \mathcal{M} is an algebra, to show it is a σ -algebra, by Lemma 1 we need only show \mathcal{M} is closed under countable disjoint unions. But if $\{A_n\}$ is a disjoint sequence in \mathcal{M} ,

$$\nu(E) = \nu\left(E \cap \left[\bigcup_1^N A_n\right]\right) + \nu\left(E \cap \left[\bigcup_1^N A_n\right]^c\right) \geq \sum_1^N \nu(E \cap A_n) + \nu\left(E \cap \left[\bigcup_1^\infty A_n\right]^c\right)$$

Take $N \rightarrow \infty$ and use (8); then

$$\nu(E) \geq \sum_1^\infty \nu(E \cap A_n) + \nu\left(E \cap \left[\bigcup_1^\infty A_n\right]^c\right)$$

This is what is needed to show $\bigcup_1^\infty A_n$ is in \mathcal{M} . Thus \mathcal{M} is a σ -algebra.

The countable additivity of ν on \mathcal{M} is a consequence of (8) with $E = S$. \diamond

6.2 Completeness and outer measures.

Here is an interesting fact about outer measures. Let A be a subset of S . If $\nu(A) = 0$, then A is ν -measurable. Indeed, if $\nu(A) = 0$ and E is any subset of A , then $\nu(E \cap A) = 0$ (by monotonicity of ν) and so, using also subadditivity,

$$\nu(E) \geq \nu(E \cap A^c) = \nu(E \cap A) + \nu(E \cap A^c) \geq \nu(E).$$

Since this is true for any subset E , A is ν -measurable.

Consider now (S, \mathcal{M}, ν) where \mathcal{M} is the σ -algebra of ν -measurable sets. By what we have just shown, if $A \in \mathcal{M}$, if $\nu(A) = 0$, and if $B \subset A$, then B is also in \mathcal{M} (and $\nu(B) = 0$).

In general, a measure space (S, Σ, ρ) is said to be **complete** if $A \in \Sigma$, $\rho(A) = 0$, and $B \subset A$ imply that $B \in \Sigma$. We have just proved.

Proposition 4 *If ν is an outer measure on the subsets of S and if \mathcal{M} is the σ -algebra of ν -measurable sets, then (S, \mathcal{M}, ν) is complete.*

Any measure space (S, Σ, ρ) has a completion. Let \mathcal{N} be the subsets N of S such that there exists an M in Σ with $N \subset M$ and $\rho(M) = 0$. Define $\bar{\Sigma}$ as follows: a subset A of S is in $\bar{\Sigma}$ if there exists an $A' \in \Sigma$ and an $N \in \mathcal{N}$ such that $A = A' \cup N$. For such an A and A' , define $\rho(A) = \rho(A')$.

Proposition 5 *$(S, \bar{\Sigma}, \rho)$ is complete*

Proof. Let A, A', N , and M be such that $A = A' \cup N$, $N \subset M$, $A' \in \Sigma$, $M \in \Sigma$, and $\rho(M) = 0$. Thus A is in $\bar{\Sigma}$. Now

$$A^c = (A')^c \cap N^c = [(A')^c \cap M^c] \cup [N^c \cap M].$$

and this shows that A^c is also in $\bar{\Sigma}$. The proof that $\bar{\Sigma}$ is closed under countable unions is left as an exercise. With this, it follows that $\bar{\Sigma}$ is a σ -algebra.

It is necessary to show that ρ is consistently defined on $\bar{\Sigma}$. Thus, let $A = A' \cup N = A'' \cup N'$, where now A'' is also in Σ and $N' \subset M'$ for a set $M' \in \Sigma$ of measure zero. We to show that $\rho(A') = \rho(A'')$. Since $A'' = A \cap A'' = (A' \cap A'') \cup (N \cap A'')$ and since also $A'' = (A' \cap A'') \cup (A'' \cap (A')^c)$, it follows that $A'' \cap (A')^c \subset N \cap A'' \subset M$. Hence $\rho(A'') = \rho(A' \cap A'')$. Reversing the roles of A' and A'' similarly gives $\rho(A') = \rho(A' \cap A'')$.

The countable additivity of ρ is straightforward and is left as an exercise. \diamond

The concept of completion allows clarification of the relation between $\sigma(\mathcal{R})$ and \mathcal{M} in Theorem 2.

Theorem 3 *Let $(S, \mathcal{R}, \overset{\circ}{\mu})$ be as in Theorem 2, let $\overset{\circ}{\mu}$ be continuous from below, and let (S, \mathcal{M}, μ) be the extension of $\overset{\circ}{\mu}$ provided by Theorem 2; $(\mu(A) \triangleq \mu^*(A) \text{ for } A \in \mathcal{M}.)$ If $\overset{\circ}{\mu}$ is σ -finite, then (S, \mathcal{M}, μ) is the completion of $(S, \sigma(\mathcal{R}), \mu)$.*

The proof is left as an exercise.

6.3 Proof of Theorem 2

Return to the situation of Theorem 2. We have a finitely additive measure $\overset{\circ}{\mu}$ on (S, \mathcal{R}) , where \mathcal{R} is an algebra, the outer measure μ^* induced by $\overset{\circ}{\mu}$, and the collection \mathcal{M} of μ^* -measurable sets. Proposition 3 implies that \mathcal{M} is a σ -algebra and μ^* , restricted to \mathcal{M} is a measure. This is essentially the claim of Theorem 2, part (b).

To prove Theorem 2, part (a), we must show that $\mathcal{R} \subseteq \mathcal{M}$. In other words, for any set $A \in \mathcal{R}$, we must show that

$$\mu^*(E) = \mu^*(E \cap A) + \mu^*(E \cap A^c) \quad \text{for all } E \subseteq S. \quad (9)$$

As usual, it suffices to prove the left-hand side of (9) is greater than or equal to the right-hand side. Let $\{E_n\}_{n \geq 1}$ be any sequence of sets in \mathcal{R} which covers E . Then $\{E_n \cap A\}_{n \geq 1}$ and $\{E_n \cap A^c\}_{n \geq 1}$ are sequences in \mathcal{R} that cover $E \cap A$ and $E \cap A^c$ respectively. Therefore,

$$\mu^*(E \cap A) \leq \sum_1^\infty \overset{\circ}{\mu}(E_n \cap A), \quad \text{and} \quad \mu^*(E \cap A^c) \leq \sum_1^\infty \overset{\circ}{\mu}(E_n \cap A^c).$$

By the finite additivity of $\overset{\circ}{\mu}$ on \mathcal{R} ,

$$\sum_1^\infty \overset{\circ}{\mu}(E_n) = \sum_1^\infty \overset{\circ}{\mu}(E_n \cap A) + \overset{\circ}{\mu}(E_n \cap A^c) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c).$$

Since $\mu^*(E)$ is the infimum of the left-hand side over all covers of E by sets in \mathcal{R} , $\mu^*(E) \geq \mu^*(E \cap A) + \mu^*(E \cap A^c)$, which is what we needed to prove.

To prove part (c) of Theorem 2, we must show that if $\overset{\circ}{\mu}$ is continuous from below on \mathcal{R} , then for every $A \in \mathcal{R}$, $\mu^*(A) = \overset{\circ}{\mu}(A)$. Certainly $\mu^*(A) \leq \overset{\circ}{\mu}(A)$, by definition of $\mu^*(A)$. To show the opposite inequality, we must show that if $A \subseteq \cup_1^\infty E_n$, where each E_n is in \mathcal{R} , then

$$\overset{\circ}{\mu}(A) \leq \sum_1^\infty \overset{\circ}{\mu}(E_n). \quad (10)$$

For each n , let $B_n \triangleq E_n - \cup_1^n E_i$. Then $\{A \cap B_n\}_{n \geq 1}$ is a disjoint sequence of sets in \mathcal{R} and $A = \cup_1^\infty (A \cap B_n)$. By Lemma 4 (b),

$$\overset{\circ}{\mu}(A) = \sum_1^\infty \overset{\circ}{\mu}(A \cap B_n).$$

But since $A \cap B_n \subseteq E_n$ for each n , this implies (10).

Finally, it remains to prove the claim of uniqueness in part (d). Let μ denote the outer measure restricted to \mathcal{M} . Suppose that ν is a measure on (S, \mathcal{M}) . Since $\overset{\circ}{\mu}$ is assumed to be σ -finite, there is a sequence $\{K_n\}_{n \geq 1}$ of sets in \mathcal{R} such that $\mu(K_n) (= \overset{\circ}{m}(K_n)) < \infty$ for all n , and $S = \cup_1^\infty K_n$. It may be assumed that the K_n are disjoint. Let $\mathcal{R} \cap K_n$ denote the algebra of sets in \mathcal{R} that are contained in K_n . Since μ and ν agree on $\mathcal{R} \cap K_n$ and since the collection of subsets of $\mathcal{R} \cap K_n$ on which μ and ν agree is a monotone class (use the properties of continuity from above and below of measures), μ and ν agree on $\sigma(\mathcal{R} \cap K_n)$ by the monotone class theorem. For each n $\sigma(\mathcal{R} \cap K_n) = \sigma(\mathcal{R}) \cap K_n$ (see below). Therefore, for any $A \in \sigma(\mathcal{R})$,

$$\mu(A) = \sum_1^\infty \mu(A \cap K_n) = \sum_1^\infty \nu(A \cap K_n) = \nu(A).$$

This shows that μ and ν agree on $\sigma(\mathcal{R})$.

Now let $A \in \mathcal{M}$, and assume that $\mu(A) < \infty$. From Theorem 3, there are sets A' and M in \mathcal{R} and a set $N \in \mathcal{M}$ with $A = A' - N$, $N \subseteq M$, and $\mu(M) = 0$. It follows that $\nu(N) = 0$ as well because $\nu(M) = \mu(M) = 0$. Hence $\nu(A) = \nu(A') = \mu(A)$. For a general $A \in \mathcal{M}$, $\nu(A) = \sum_1^\infty \nu(A \cap K_n) = \sum_1^\infty \mu(A \cap K_n) = \mu(A)$. Thus we have shown ν and μ agree on \mathcal{M} .

Finally, for any fixed n , consider the σ -algebras $\sigma(\mathbb{R} \cap K_n)$ and $\sigma(\mathcal{R} \cap K_n^c)$. Consider the collection of all sets of the form $A \cup B$, where $A \in \sigma(\mathcal{R} \cap K_n)$ and $B \in \sigma(\mathcal{R} \cap K_n^c)$. It is easy to confirm that this collection is a σ -algebra that contains \mathcal{R} . Hence it equals \mathcal{R} . This can only happen if $\sigma(\mathbb{R} \cap K_n) = \sigma(\mathcal{R}) \cap K_n$ and $\sigma(\mathbb{R} \cap K_n^c) = \sigma(\mathcal{R}) \cap K_n^c$.

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