501: Proof of the Fubini-Tonelli theorem

These notes sketch a proof of the Fubini-Tonelli theorem in the setting of Lebesgue measure on $\mathbb{R}^{n+k}$. It is similar in spirit to the proof in the text, but concentrates first on the Borel measurable case; also it invokes the monotone class theorem—see problem 32.

We first prove the theorem for non-negative, Borel measurable functions. The following notation is needed. For a function f defined on $\mathbb{R}^{n+k}$ and a point $x \in \mathbb{R}^n$, define the *section* $f(x, \cdot)$ of f at x to be the function $y \in \mathbb{R}^k \to f(x, y)$. Define the section $f(\cdot, y)$ for a fixed point $y \in \mathbb{R}^k$ similarly. If A is a subset of $\mathbb{R}^{n+k}$, define the section of A at x, $x \in \mathbb{R}^n$, to be the subset A_x of $\mathbb{R}^k$, defined by

$$A_x = \{y\,;\,(x,y) \in A\}\,.$$

Define A^y similarly as the section of A at a point y in $\mathbb{R}^k$. Notice that $\chi_{A_x}(y) = \chi_A(x, y)$; in other words, $\chi_{A_x}(\cdot) = \chi_A(x, \cdot)$.

A *rectangle* in $\mathbb{R}^m$ means specifically a set of the form

$$(a_1, b_1] \times \cdots \times (a_m, b_m].$$

The algebra of all finite disjoint unions of rectangles in $\mathbb{R}^m$, shall be denoted $\mathcal{V}^{(m)}$.

As a preliminary step, we show that Fubini-Tonelli is valid for indicator function of rectangles. This is a simple consequence of the definitions. Consider a rectangle $U = (a_1, b_1] \times \cdots \times (a_{n+k}, b_{n+k}]$ in $\mathbb{R}^{n+k}$. Then $U = V \times W$, where V is the rectangle $(a_1, b_1] \times \cdots \times (a_n, b_n]$ and W is the rectangle $(a_{n+1}, b_{n=1}] \times \cdots \times (a_{n+k}, b_{n+k}n]$. Therefore,

$$\chi_U(x, \cdot) = \begin{cases} \chi_W(y), & \text{if } x \in V; \\ 0, & \text{otherwise,} \end{cases} \qquad \text{or, equivalently,} \qquad U_x = \begin{cases} W, & \text{if } x \in V; \\ \emptyset, & \text{otherwise.} \end{cases}$$

It follows immediately that

$$\chi_U(x, \cdot) \text{ is a Borel measurable function on } \mathbb{R}^k \text{ for each } x \in \mathbb{R}^n. \tag{1}$$

(Equivalently, U_x is a Borel subset of $\mathbb{R}^k$ for each $x \in \mathbb{R}^n$.) Also, it follows that

$$\int_{R^k} \chi_U(x, y)\, dm_k(y) = m_k(U_x) = m_k(W)\chi_V(x). \tag{2}$$

and thus

$$x \to \int_{R^k} \chi_U(x, y)\, dm_k(y) \quad \text{define a Borel measurable function on } R^n. \tag{3}$$

Finally, by definition, the Lebesgue measure of a rectangle is its volume, the product of the lengths of its sides, and thus, from (2):

$$m_{n+k}(U) = m_n(V)m_k(W) = \int_{R^n} m_k(W)\chi_V(x)\, dm_n(x) = \int_{R^n} \left[\int_{R^k} \chi_U(x, y)\, dm_k(y)\right] dm_n(x). \tag{4}$$

Statements (1), (3), and (4) are essentially the statements of the Fubini-Tonelli theorem when applied to rectangles. The following lemma states that properties (1),(3), (4) generalize to arbitrary Borel subsets.

Lemma 1 *Every Borel measurable subset A of $\mathbb{R}^{n+k}$ satisfies*

(i) *A_x is a Borel measurable subset of $\mathbb{R}^k$ for every $x \in \mathbb{R}^n$;*

(ii) *The function* $x \to \int_{R^k} \chi_A(x,y)\, dm_k(y) = m_k(A_x)$ *is Borel measurable.*

(iii) $$m_{n+k}(A) = \int_{R^{n+k}} \chi_A(z)\, dm_{n+k}(z) = \int_{R^n} \left[\int_{R^k} \chi_A(x,y)\, dm_k(y) \right] dm_n(x).$$

Similar statement also hold for A^y and $m_{n+k}(A) = \int_{R^k} \left[\int_{R^n} \chi_A(x,y)\, dm_n(x) \right] dm_k(y)$.

Proof: For any $N > 0$, let $\mathcal{C}_N$ be the collection of all Borel subsets A of $\mathbb{R}^{n+k}$ such that $A \cap (-N, N]^{n+k}$ satisfies properties (i), (ii), and (iii). By (1),(3), and (4), $\mathcal{C}_N$ contains all rectangles. Clearly $\mathcal{C}$ is also closed under finite disjoint unions, and therefore $\mathcal{C}$ contains the algebra $\mathcal{V}^{n+k}$. Next, observe that $\mathcal{C}$ is also a monotone class. For example, if $A_j \uparrow A$, then for every x $((-N, N]^{n+k} \cap A_j)_x \uparrow ((-N, N]^{n+k} \cap A)_x$, so if $((-N, N]^{n+k} \cap A_j)_x$ is a Borel set for every j, so is $((-N, N]^{n+k} \cap A_j)_x$. This shows that property (i) is preserved under increasing limits. A similar argument works for decreasing limits. The closlure of properties (ii) and (iii) under increasing and decreasing limit is proved by using the fact that suprema and infima of countable families of Borel measurable functions are Borel and the dominated convergence theorem. The reason for bounding by intersection with the finite measure set $(-N, N]^{n+k}$ in this step is to be able to use the dominated convergence theorem to prove the closure of property (iii) under decreasing limits.

By the monotone class theorem, we can now conclude $\mathcal{C}_N$ equals the Borel σ-algebra of $\mathbb{R}^{n+k}$ for every positive, finite N.

Let A be any Borel set of $\mathbb{R}^{n+k}$. Since $A \cap (-N, N]^{n+k} \uparrow A$ as $N \to \infty$, and $A \cap (-N, N]^{n+k}$ satisfies (i), (ii), (iii), we can again easily conclude the same properties hold for A; for property (iii) we use the monotone convergence theorem. $\diamond$

Given Lemma 1, it is clear that for any Borel measurable, simple function f:

(a) $f(x, \cdot)$ is a Borel measurable function on $\mathbb{R}^k$ for every x in $\mathbb{R}^k$. Likewise $f(\cdot, y)$ is a Borel function on $\mathbb{R}^n$ for every y in $\mathbb{R}^n$.

(b) $x \to \int_{R^k} f(x,y)\, dm_k(y)$ is a Borel measurable function of x, and likewise for $y \to \int_{R^n} f(x,y)\, dm_n(x)$.

$$\text{(c)} \quad \int_{\mathbb{R}^{n+k}} f(z)\, dm_{n+k}(z) = \int_{R^n} \left[\int_{R^k} f(x,y)\, dm_k(y) \right] dm_n(x)$$
$$= \int_{R^k} \left[\int_{R^n} f(x,y)\, dm_n(x) \right] dm_k(y)$$

It is clear that for non-negative functions, each of the properties (a), (b), (c) is closed under increasing limits; that is, if $0 \leq f_1 \leq f_2 \leq \cdots$, and each f_n satisfies (a), (b), (c), then so does $f = \lim_n f_n$; for example, for property (c), use the Monotone Convergence Theorem. Since every non-negative, Borel measurable function is an increasing limit of a sequence of non-negative simple functions, it follows that:

Proposition 1 *Every non-negative, Borel measurable function satisfies (a), (b), (c).*

This proposition is the statement of the Fubini-Tonelli theorem for non-negative, Borel measurable functions. It remains to treat functions with both positive and negative values. To do this it is convenient to introduce the concept of an *almost everywhere (a.e.) defined function*. A function defined on a subset D of $\mathbb{R}^d$ is said to be a.e. defined if the Lebesgue measure of the complement of D is zero. An a.e. defined function g is Borel measurable if its domain D is Borel measurable and g is Borel measurable as a function defined on D. One defines similarly Lebesgue measurable, a.e. defined functions. Since the integrals of two functions differing on a set of measure zero are the same, the definition of the Lebesgue integral can be extended to a.e. defined functions. If g is a.e. defined on $\mathbb{R}^d$ and either Lebesgue or Borel measurable, define $\bar{g}(x) = g(x)$ if x is in the domain of g, and $g(x) = 0$, if not—there is no significance to choosing 0 as the value of g on the complement of D; any extension of g to a measurable function on $\mathbb{R}^d$ will do. We say g is integrable if $\bar{g}$ is, and we define $\displaystyle\int g\, dm_d \triangleq \int \bar{g}\, dm_d$.

Now let f be a Borel measurable function on $\mathbb{R}^{n+k}$, and let f^+ and f^- be its postive and negative parts, so that $f = f^+ - f^-$. Proposition 1 applies to f^+ and to $f-$ individually. Since $f(x,\cdot) = f^+(x,\cdot) - f^-(x,\cdot)$, we see at once that *property (a) is true for every Borel measurable function.*

By proposition 1, the functions

$$x \to \int_{R^k} f^+(x,y)\, dm_k(y) \quad \text{and} \quad x \to \int_{R^k} f^-(x,y)\, dm_k(y)$$

are Borel measurable functions with values in $[0,\infty]$. Let

$$D \triangleq \left\{ x\, ;\, \int_{R^k} f^+(x,y)\, dm_k(y) \neq \infty \right\} \cup \left\{ x\, ;\, \int_{R^k} f^-(x,y)\, dm_k(y) \neq \infty \right\}.$$

Then for every $x \in D$, $\displaystyle\int_{R^k} f(x,y)\, dm_k(y)$ is defined (in the extended reals), and

$$\int_{R^k} f(x,y)\, dm_k(y) = \int_{R^k} f^+(x,y)\, dm_k(y) - \int_{R^k} f^-(x,y)\, dm_k(y). \tag{5}$$

Since D is a Borel measurable set and both functions on the right-hand side of (5) are Borel measurable,

$$x \to \int_{R^k} f(x,y)\, dm_k(y) \quad \text{defines a Borel measurable function on } D.$$

With these observations, we can prove the Fubini-Tonelli theorem for general, Borel measurable functions on $\mathbb{R}^{n+k}$.

Proposition 2 *Let f be Borel measurable and suppose $f \in L^1(\mathbb{R}^{n+k})$ (that is, $\int |f|\, dm_{n+k} < \infty$). Then*

(i) Every section $f(x,\cdot)$, $x \in \mathbb{R}^n$, and every section $f(\cdot,y)$, $y \in \mathbb{R}^k$, is Borel measurable.

(ii) $x \to \int_{R^k} f(x,y)\, dm_k(y)$ and $y \to \int_{R^n} f(x,y)\, dm_n(x)$ are a.e. defined, Borel measurable, integrable functions.

(iii) $$\int_{\mathbb{R}^{n+k}} f(z)\, dm_{n+k}(z) = \int_{R^n} \left[\int_{R^k} f(x,y)\, dm_k(y)\right] dm_n(x) = \int_{R^k} \left[\int_{R^n} f(x,y)\, dm_n(x)\right] dm_k(y)$$

Proof: We have already seen that (i), which is just the restatement of property (a), is true.

By applying Proposition 1 to $|f|$ and using integrability

$$\int_{R^n} \left[\int_{R^k} |f|(x,y)\, dm_k(y)\right] dm_n(x) = \int_{\mathbb{R}^{n+k}} f(z)\, dm_{n+k}(z) < \infty.$$

It follows that $\int_{R^k} |f|(x,y)\, dm_k(y) < \infty$ for a.e. x, and since $0 \le f^+ \le |f|$ and $0 \le f^- \le |f|$, we see that $m_n(D^c) = 0$, where D is the domain defined in the paragraph preceding Proposition 2. It also follows that $x \to \int_{R^k} f^+(x,y)\, dm_k(y)$ and $x \to \int_{R^k} f^-(x,y)\, dm_k(y)$ The discussion preceding Proposition 2 then implies that $x \to \int_{R^k} f(x,y)\, dm_k(y)$ is an a.e. defined, Borel measurable, integrable function, proving statement (ii).

To prove statement (iii), use $f = f^+ - f^-$, apply the multiple integration formula of property (c) to f^+ and f^- individually, and use linearity of the integral. ⋄

Finally, it remains to establish the Fubini-Tonelli theorem for Lebesgue integrable functions. But if we can establish an extension of Lemma 1 to Lebesgue measurable functions, we can argue as we just did, proceding from sets to simple functions to positive functions, and finally to general, Lebesgue measurable functions. Thus it will suffice to prove the following.

Lemma 2 *Every Lebesgue measurable subset A of $\mathbb{R}^{n+k}$ satisfies*

(i) A_x is a Borel measurable subset of $\mathbb{R}^k$ for a.e. every $x \in \mathbb{R}^n$;

(ii) The a.e. function $x \to \int_{R^k} \chi_A(x,y)\, dm_k(y) = m_k(A_x)$ is Lebesgue measurable.

(iii) $$m_{n+k}(A) = \int_{R^{n+k}} \chi_A(z)\, dm_{n+k}(z) = \int_{R^n} \left[\int_{R^k} \chi_A(x,y)\, dm_k(y) \right] dm_n(x).$$

Similar statement also hold for A^y and $m_{n+k}(A) = \int_{R^k} \left[\int_{R^n} \chi_A(x,y)\, dm_n(x) \right] dm_k(y)$.

Notice, that it is not true that every section of a Lebesgue set is necessarily Lebesgue measurable, but a.e. section will be, and this is enough.

Proof: Let A be a Lebesgue measurable set of $\mathbb{R}^{n+k}$. We know there exist Borel measurable sets F and G such that $F \subseteq A \subseteq G$ and $m_{n+k}(G \setminus F) = 0$. Observe that for every x, $(G \setminus F)_x = G_x \setminus F_x$ and $F_x \subset A_x \subset G_x$. Let D denote the set of x where $m_k((G \setminus F)_x) = 0$. It follows from Lemma 1 that $m((G \setminus F)_x) = 0$ for a.e. x, that is, that $m_n(D^c) = 0$. But for $x \in D$, A_x differs from the Borel measurable sets F and G by a set of outer measure zero and hence A_x is Lebesgue measurable. This proves (i).

Consider the function $x \to m_k(A_x)$, whose domain is the set of x for which A_x is Lebesgue measurable. Then $m_k(A_x)$ is defined and equals $m(G_x)$ on the set D. Thus $x \to m(A_x)$ differs from the Borel measurable function $x \to m(G_x)$ at most on a set of measure zero, and hence it is Lebesgue measurable, proving (ii).

Finally,

$$m_{n+k}(A) = m_{n+k}(G) = \int_{R^n} m_k(G_x)\, dm_n(x) = \int_{\mathbb{R}^n} m_k(A_x)\, dm_n(x),$$

which proves (iii). ◇

For completeness, here is the general statement of the Fubini-Tonelli theorem for the Lebesgue measurable case. Let (a') and (b') be the same as the previously defined statments (a) and (b) except that "Borel measurable" and "for every" are replaced by "Lebesgue measurable" and "for a.e.", and in (b') the function is defined only a.e. Let (i') and (ii') be the analogous statements of Proposition 2, with the same changes.

Theorem 1 *Any positive, Lebesgue measurable function satisfies (a'), (b'), and (c). Any integrable, Lebesgue measurable function satisfies the modified statements (i') and (ii') of Proposition 2 and statement (iii) of Proposition 2.*