Problems I; Real Analysis, 640:501, Fall, 2003

Problems 1-9 are on topics from undergraduate real analysis and elementary point set topology—limits inferior and superior, compactness and continuity. For compactness and continuity on $\mathbb{R}^n$, most of what you need to know is covered in Chapter 1 of Wheeden and Zygmund. Some of the problems below are stated in the general setting of a metric space, which is not treated in Wheeden and Zygmund's review. If you are uncomfortable with this, just substitute $\mathbb{R}^n$. An excellent source for review of topology and analysis basics for this course is W. Rudin, *Principles of Mathematical Analysis*, Chapters 2-4. Rudin does not treat general topological spaces. For this G. Folland, *Real Analysis: Modern Techniques and Their Applications*, Chapter 4, is a concise treatment. A classic first text in topology is Munkres, *Topology*; Chapters 2-4, the statement and proof of Tychonoff's theorem, and Chapter 7 on complete metric spaces are a minimum of what "every mathematician should know" about point-set topology.

Some problems below can be solved using the results of previous problems. Feel free to use a previous problem, even if you haven't solved it.

In all problems below, we allow $+\infty$ and $-\infty$ as values of limiting operations. The notation $\mathbb{R}$ is used for the real numbers, and $\overline{\mathbb{R}}$ for the extended reals, that is $\mathbb{R}$ with the points $-\infty$ and $+\infty$ adjoined.

1. Let $\{x_n\}$ be a sequence of real numbers. Show that $\{x_n\}$ contains a subsequence converging to $\liminf_{n\to\infty} x_n$ and a subsequence converging to $\limsup_{n\to\infty} x_n$.

2. Let $\{x_n\}$ be a sequence of real numbers, and let $a_n := (x_1 + \cdots + x_n)/n$.

i). Show that

$$\liminf_{n\to\infty} x_n \leq \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \leq \limsup_{n\to\infty} x_n.$$

ii). Show that if $\lim_{n\to\infty} x_n = L$, then $\lim_{n\to\infty} a_n = L$.

iii). Given an example in which all limits in i) are finite and all inequalities are strict.

3. Let $\{x_n\}$ be a bounded sequence of real numbers. Suppose that there exists a number L such that, for any *convergent* subsequence $\{x_{n_k}\}$ of $\{x_n\}$, $\lim_{k\to\infty} x_{n_k} = L$. ($L = \pm\infty$ is allowed and a sequence with $\lim_{n\to\infty} y_n = \pm\infty$ is considered to be convergent.) Does it follow that $\lim_{n\to\infty} x_n = L$? Prove or give a counterexample.

4. Consider the set of rational numbers as a metric space with the distance metric $d(p,q) = |p-q|$. Let A be the subset of rational numbers p with $2 < p^2 < 3$. Show that A is closed and bounded, but not compact. (Of course, A is not compact because it isn't complete. However, show that A is not compact directly, by exhibiting an open cover of A without a finite subcover.)

5. Define the distance between two subsets A and B of $\mathbb{R}^n$ by $d(A,B) = \inf\{|x-y| \mid x \in A, y \in B\}$.

i). Give an example of two disjoint closed sets A and B such that $d(A,B) = 0$.

ii). Show that if A and B are closed and A is compact, then $d(A,B) > 0$.

iii). Show that for any subset A of $\mathbb{R}^n$, $x \longrightarrow d(x,A)$ is continuous in x.

Problems 6-9 explore the concept of upper semicontinuity. A function $f : X \longrightarrow \overline{\mathbb{R}}$ on a metric space X is called *upper semicontinuous* if for every $x \in X$,

$$\limsup_{n\to\infty} f(x_n) \leq f(x) \quad \text{whenever} \quad \lim_{n\to\infty} x_n = x. \tag{1}$$

Similarly, f is said to be *lower semicontinuous* if

$$\liminf_{n\to\infty} f(x_n) \geq f(x) \quad \text{whenever} \quad \lim_{n\to\infty} x_n = x. \tag{2}$$

6. Let $f : X \longrightarrow \overline{\mathbb{R}}$ be a function on the metric space X. Show that f is upper semicontinuous if and only if the set $f^{-1}([a,\infty])$ is closed for every real number a.

7. Semicontinuity is a useful property in studies of optimization. Here is why. Show that if X is a compact metric space and $f : X \to \overline{\mathbb{R}}$ is upper semicontinuous, there is a point $x^* \in X$ at which f achieves its maximum, that is

$$f(x^*) \geq f(x) \quad \text{for all } x \in X.$$

8. Let $\{f_\alpha \mid \alpha \in \mathcal{A}\}$, where $\mathcal{A}$ is some index set, be a family of functions from a metric space X to $\overline{\mathbb{R}}$. Prove: If f_α is upper semicontinuous for every $\alpha \in \mathcal{A}$, then $f(x) := \inf\{f_\alpha(x) \mid \alpha \in \mathcal{A}\}$, $x \in X$, defines an upper semicontinuous function.

9. Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. In this problem we are interested in studying the set of points at which f may be discontinuous. To this end, define the function

$$\omega_f(x, \delta) := \sup_{[x-\delta, x+\delta]} f(x) - \inf_{[x-\delta, x+\delta]} f(x), \qquad x \in \Re, \quad \delta > 0.$$

For each x and δ, this gives the *oscillation* of f on the interval $[x - \delta, x + \delta]$. Show first (easy) that $\omega_f(x, \delta)$ is increasing in δ, and hence that

$$\omega_f(x) := \inf\{\omega_f(x, \delta) \mid \delta > 0\} = \lim_{\delta \to 0^+} \omega_f(x, \delta)$$

is well-defined. The quantity $\omega_f(x)$ is the *oscillation of f at the point x.* Show that f is continuous at a point x if and only if $\omega_f(x) = 0$ (this is simply a matter of chasing definitions).

Now here is the interesting point, which places some restrictions on the nature of the set of discontinuities of an arbitrary $f : \mathbb{R} \to \mathbb{R}$. *The function $\omega_f(x)$ is upper semicontinuous.* Prove this. (Hint: Problem 6 is not so useful; $\omega_f(x, \delta)$ for fixed $\delta > 0$ is not necessarily upper semicontinuous. It may be helpful to use that $\omega_f(x, \delta) \geq \omega_f(z, \delta/2)$ for all z such that $|z - x| \leq \delta/2$.)

Discussion: Observe that

$$\{x \mid f \text{ is not continuous at } x\} = \{x \mid \omega_f(x) > 0\} = \cup_{n=1}^{\infty} \{x \mid \omega_f(x) \geq \frac{1}{n}\}.$$

By the upper semicontinuity of ω_f, it follows that the set of discontinuity points of f is a countable union of closed sets. Such a set is called an F_σ set. Thus, the set of discontinuity points of a function cannot be arbitrary.

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10. Construct a function on $[0, 1]$ which is increasing and which has a discontinuity at every rational point in $(0, 1)$.

11. (Wheeden and Zygmund, p. 31, problem 4.) Let $\{f_n\}$ be a sequence of functions on $[a,b]$, and assume $f(x) \triangleq \lim_n f_n(x)$ exists and is finite for each $x \in [a,b]$. If $V[f_n; a,b] \leq M < \infty$ for each n, then $V[f;a,b] \leq M$.

12. (Wheeden and Zygmund, p. 31, problem 5.) Let $f : [a,b] \to \mathbb{R}$. If $V[f; a+\epsilon, b] \leq M < \infty$ for every $\epsilon > 0$. Then $V[f;a,b] < \infty$. Is $V[f,a,b] \leq M$ in all cases? If not, what additional assumption will make it so?

13. (Wheeden and Zygmund, p. 31, problem 6.) Show that $V[g; 0, 1] < \infty$, where $g(x) = x^2 \sin(1/x)$ if $0 < x$, and $g(0) = 0$.

14. Let $f : [a,b] \to \mathbb{R}$ be a function of bounded variation and let x be a point in $[a,b]$. f is continuous at x if and only if $V_f[a,x]$ is continuous at x.

15. (Try this problem before looking at the proof of Theorem 2.30, page 30, of Wheeden and Zygmund.) Let L_P and U_P denote the lower and upper *Riemann sums* for partition P and function f on interval $[a,b]$. Show that if $\sup_P L_P = \inf_P U_P < \infty$, then f is Riemann integrable on $[a,b]$.

16. Let L_P and U_P denote the lower and upper *Riemann sums* for partition P and function f on interval $[a,b]$. It follows from problem 8 that the existence of a sequence of partitions $\{P_n\}$ such that $\lim_n U_{P_n} - L_{P_n} = 0$ implies that f is Riemann integrable on $[a,b]$. Use this to give a direct proof that f is Riemann integrable on $[a,b]$ if $V[f;a,b] < \infty$.

17. Let f and g be Riemann integrable on $[a,b]$. Then fg is also Riemann integrable on $[a,b]$.

18. Let $\phi(x)$ be continuously differentiable on $[a,b]$. ($\phi'(a)$ is defined to be the right derivative, $\phi'(b)$ the left derivative, and ϕ' is continuous on $[a,b]$.) If $\int_a^b f\, d\phi$ exists, then

$$\int_a^b f\, d\phi = \int_a^b f \phi'\, dx.$$

19. (From problem 15, page 32, Wheeden and Zygmund) Let f be continuous and let ϕ be of bounded variation on $[a,b]$. Define $g(x) \triangleq \int_a^x f\, d\phi$ for $x \in [a,b]$. Show that $V[g;a,b] < \infty$.

20. *A continuous, nowhere differentiable function.* Later in the course, we shall see that the set of points on which a function of bounded variation is not differentiable is small in a precise sense—it has Lebesgue measure zero. Meanwhile, in this problem we present a function which is nowhere differentiable and ask you to show directly that its variation on any non-trivial interval is infinite. Examples of continuous, nowhere differentiable functions go back to Weierstrass. The example of this problem traces back to Takagi (1903) and van der Waerden (1930)—see J.B. Brown and G. Kozlowski, *Smooth Interpolation, Hölder Continuity, and the Takagi-van der Waerden Function,* **The American Mathematical Monthly**, **110**, 2003, 142-147. We follow the treatment on page 154 of W. Rudin, **Principles of Mathematical Analysis**, third edition. Let $\phi(x) \triangleq |x|$ for $|x| \leq 1$. Extend the domain of ϕ to $\mathbb{R}$ by requiring it to have period 2 ($\phi(x+2) = \phi(x)$ for all x). Define

$$f(x) \triangleq \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n \phi(4^n x).$$

This is well-defined and continuous function; indeed, since $|\phi(x)| \leq 1$ for all x, the infinite series defining f converges uniformly in x.

The function f is nowhere differentiable. The technical reasons behind this fact are as follows. First, the function $x \to \phi(4^n x)$ is differentiable except at the discrete set of points $\{m/4^n \ : \ m \text{ is an integer}\}$ and, its derivative, where it exists, is either 4^n or -4^n. Thus

$$|\phi(4^n(x+h)) - \phi(4^n x)| \leq 4^n h \quad \forall x, \ \forall h. \tag{3}$$

So, as n gets larger the functions $\phi(4^n x)$ get progressively "rougher." Second, because ϕ has period 2, $\phi(4^n x)$ has a periodicity of $2/4^n$. Therefore if $\delta = (1/2)4^{-m}$, then, since δ is an integer multiple of $2/4^n$ for any integer $n > m$,

$$f(x \pm \delta) - f(x) = \sum_{n=1}^{m} \left(\frac{3}{4}\right)^n \left[\phi(4^n(x \pm \delta)) - \phi(4^n x)\right]. \tag{4}$$

The final technical reason is the simple identity

$$3^m - \sum_{n=1}^{m-1} 3^n = \frac{3^m}{2} + \frac{3}{2}. \tag{5}$$

As we shall see, this allows the m^{th} term to dominate the difference in (2).

Now put these facts together. We will find a sequence $\{h_m\}$ such that $\lim h_m = 0$, yet $\lim |\,(f(x+h_m) - f(x))/h_m| = \infty$, This will show that f is not differentiable at x. Set $\delta_m = (1/2)4^{-m}$. From (1),(2) and (3),

$$\begin{aligned} |\, f(x \pm \delta_m) - f(x)\,| &= |\sum_{n=1}^{m} \left(\frac{3}{4}\right)^n [\phi(4^n(x\pm\delta)) - \phi(4^n x)]\,| \\ &\geq \left(\frac{3}{4}\right)^m |\,\phi(4^m(x\pm\delta_m)) - \phi(4^m)\,| - \sum_{n=1}^{m-1} 3^n \delta_m. \end{aligned}$$

Now the interval $(4^m(x-\delta_m), 4^m(x+\delta))$ has length 1, so either the interval $(4^m x, 4^m(x+\delta_m)$ or the interval $(4^m(x-\delta_m), 4^m x)$ contains no integer. Let $h_m \triangleq \delta_m$ in the first case and let $h_m = -\delta_m$ in the second case. Suppose for concreteness that the first case holds and $h_m \triangleq \delta_m$. Then on on the interval between x and $x + h_m$, $\phi(4^m x)$ is linear with a slope of 4^m. It follows from the previous equation that

$$\frac{|\, f(x+h_m) - f(x)\,|}{|h_m|} \geq \left(\frac{3}{4}\right)^m 4^m - \sum_{n=1}^{m-1} 3^n \geq \frac{3^m+3}{2}.$$

Since this expression tends to ∞ as $m \to \infty$, the proof is complete.

Problem: Show that $V_f[a,b] = \infty$ for any a and b such that $a < b$.

21. Let $f = u - w$ where both u and w are increasing functions on $[a,b]$. Suppose that $u(s) - u(s-) > 0$ and $w(s) - w(s-) > 0$. Show that $V_f([a,s]) < V_u([a,s]) + V_w([a,s])$. Likewise, show that if $u(s+) - u(s) > 0$ and $w(s+) - w(s) > 0$, then $V_f([s,t]) < V_u([s,t]) + V_w([s,t])$, for all t s.t. $s < t \leq b$. (These results imply the simultaneous jumps will not occur in the Jordan decomposition of f.)

22.a) Prove the inequality between geometric and additive means: for $a > 0$, $b > 0$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$, where $p > 1$, $q \geq 1$,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

using concavity of the natural logarithm function.

b) Prove Young's inequality: if f is real-valued, continuous and strictly increasing on $[0,\infty)$ with $f(0) = 0$, then

$$ab \leq \int_0^a f(x)\,dx + \int_0^b f^{-1}(x)\,dx,$$

where f^{-1} is the inverse function of f. Use Young's inequality to reprove the inequality of a).

c) Take $a = \dfrac{f}{\int_c^d |f|^p\,dx}$, $b = \dfrac{g}{\int_c^d |g|^q\,dx}$ to derive Hölder's inequality:

$$\int_c^d |fg|\,dx \le \left(\int_c^d |f|^p\,dx\right)^{1/p} \left(\int_c^d |g|^q\,dx\right)^{1/q}.$$

23. A function $f : (a,b) \to \mathbb{R}$ is convex (up) if

$$f(\theta x + (1-\theta)y) \le \theta f(x) + (1-\theta) f(y) \qquad \forall\ a < x, y < b,\ \ \forall\ 0 \le \theta \le 1.$$

i). If f is convex, if $x < x'$ and if $y < y'$, then

$$\frac{f(y) - f(x)}{y - x} \le \frac{f(y') - f(x')}{y' - x'}.$$

ii). If f is convex on (a,b), $D_+ f(x) \triangleq \lim_{y \downarrow x} \dfrac{f(y) - f(x)}{y - x}$ and $D_- f(x) \triangleq \lim_{uparrowx} \dfrac{f(y) - f(x)}{y - x}$ exits for every $x \in (a,b)$ and are monotone increasing functions. (Note: these facts imply that f is continuous on (a,b).

iii). If f is convex on (a,b), then f is differentiable at all but (possibly) a countable number of points in (a,b) and $f'(x)$ is increasing in x.

24. Any $x \in [0,1]$ has a triadic expansion $x = \sum_1^\infty \omega_i 3^{-i}$, where each ω_i is either 0, 1, or 2. The expansion is unique, except if $x = k3^{-j}$ for positive integers k and j. Show that if x is in the Cantor set then x has a triadic expansion for which $\omega_i \in \{0, 2\}$, for each i.

25. The Cantor set is constructed by successivelly removing middle thirds. Instead remove a middle fraction β at each step, $0 < \beta < 1$. Show that this set is perfect. The complement of this set is a countable disjoint union of open intervals $\cup_1^\infty I_i$. If $\ell(I_i)$ is the length of interval I_i, show that $\sum \ell(I_i) = 1$.

26. a) Let C be the Cantor set in $[0,1]$. Let

$$f(x) \triangleq \begin{cases} 0, & \text{if } x \in C; \\ 1, & \text{otherwise,} \end{cases}$$

Show directly that f is Riemann integrable on $[0,1]$.