

501 Lecture Notes: Algebras and σ -algebras

1 Definitions

Let S be a nonempty set.

Definition 1 A collection \mathcal{A} of subsets of S is called an algebra if $S \in \mathcal{A}$ and \mathcal{A} is closed under the operations of taking complements, forming finite unions of elements of \mathcal{A} and forming finite intersections of elements of \mathcal{A} .

Lemma 1 Suppose the collection \mathcal{A} of subsets of S satisfies

- (i) $S \in \mathcal{A}$.
- (ii) $B \in \mathcal{A}$ implies $B^c \in \mathcal{A}$.
- (iii) $B \in \mathcal{A}$ and $C \in \mathcal{A}$ imply that $B \cap C \in \mathcal{A}$.

Then \mathcal{A} is an algebra.

Proof: Use induction and DeMorgan's laws. \diamond

Lemma 1 remains true if (iii) is replaced by [(iii')] $B \in \mathcal{A}$ and $C \in \mathcal{A}$ imply that $B \cup C \in \mathcal{A}$.

Definition 2 A collection \mathcal{A} of subsets of S is called an σ -algebra if $S \in \mathcal{A}$ and \mathcal{A} is closed under the operations of taking complements, forming countable unions of elements of \mathcal{A} and forming countable intersections of elements of \mathcal{A} .

Lemma 2 Suppose the collection \mathcal{A} of subsets of S satisfies

- (i) $S \in \mathcal{A}$.
- (ii) $B \in \mathcal{A}$ implies $B^c \in \mathcal{A}$.
- (iii) If $B_i \in \mathcal{A}$ for $i \geq 1$, then $\bigcap_1^\infty B_i \in \mathcal{A}$.

Then \mathcal{A} is a σ -algebra.

As before condition [(iii)] in Lemma 2 can be replaced by closure under countable unions.

Note also that any σ -algebra is of course an algebra. Any finite algebra is a σ -algebra.

Let f be a mapping from S into another nonempty set X . If \mathcal{X} is a collection of subsets of X , let

$$f^{-1}(\mathcal{X}) \triangleq \{f^{-1}(U); U \in \mathcal{X}\}.$$

Lemma 3 *If \mathcal{X} is an algebra, respectively σ -algebra, of X , then $f^{-1}(\mathcal{X})$ is an algebra, respectively a σ -algebra, of subsets of S .*

Proof. The lemma is an immediate consequence of the following identities, which are easy to derive from the definitions of inverse image, union, and intersection for any function f and subsets A, A_1, A_2, \dots of the range of f .

$$f^{-1}\left(\bigcup_{\lambda \in \Lambda} A_\lambda\right) = \bigcup_{\lambda \in \Lambda} f^{-1}(A_\lambda) \quad (1)$$

$$f^{-1}\left(\bigcap_{\lambda \in \Lambda} A_\lambda\right) = \bigcap_{\lambda \in \Lambda} f^{-1}(A_\lambda) \quad (2)$$

$$f^{-1}(U^c) = (f^{-1}(U))^c \quad (3)$$

2 Simple Examples

In the following S represents an arbitrary, nonempty set.

1. The power set $\mathcal{P}(S)$ of S is a σ -algebra.
2. (The trivial σ -algebra) $\{S, \emptyset\}$ is a σ -algebra.
3. Let A_1, \dots, A_N be a disjoint partition of S . The collection of all finite unions of the sets A_1, \dots, A_N is an algebra, (and, because it is finite, also a σ -algebra).

Likewise, if $\{A_i; i \in \mathbb{N}\}$ is any countable collection of subsets of S that forms a disjoint partition of S , the collection of all countable unions of the sets A_i is a σ -algebra.

4. A right half-open rectangle in \mathbb{R}^d is a set of the form

$$\{[a_1, b_2) \times [a_2, b_2) \times \cdots \times [a_d, b_d)\} \cap \mathbb{R}^d,$$

where for each i , $1 \leq i \leq d$, $-\infty \leq a_i \leq b_i \leq \infty$. (Adding the intersection with \mathbb{R}^d in this definition may look strange. It is done to handle the case when at least one of the a_i endpoints is $-\infty$; the notation $[a_i, b_i)$ would include a point with coordinate i equal to $-\infty$ in the rectangle if we did not intersect with \mathbb{R}^d .) When $a_i = b_i$ for at least one i , interpret the rectangle to be the empty set. Let \mathcal{H} be the collection of all right half-open rectangles. It is easy to establish

$$\mathcal{H} \text{ is closed under finite intersections;} \quad (4)$$

$$\text{if } V \in \mathcal{H}, \text{ then } V^c \text{ is a finite disjoint union of sets in } \mathcal{H}. \quad (5)$$

Proposition 1 *Let \mathcal{A} be the collection of all finite disjoint unions of sets in \mathcal{H} . Then \mathcal{A} is an algebra.*

Proof: We check conditions (1), (ii), and (iii) of Lemma 1; here S is \mathbb{R}^d . First $\mathbb{R}^d \in \mathcal{H}$ by definition, so it is certainly in \mathcal{A} .

Next we establish (iii). Actually, before checking (ii) we need to know that \mathcal{A} is closed under finite intersections, but this follows from (iii) by induction. Let $B = \cup_1^n B_i$ where each B_i is in \mathcal{H} and B_1, \dots, B_n are disjoint, and let $C = \cup_1^m C_j$, where each C_j is in \mathcal{H} and C_1, \dots, C_m are disjoint. Then

$$B \cap C = \bigcup_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} B_i \cap C_j.$$

By property (4) of \mathcal{H} , $B_i \cap C_j$ is in \mathcal{H} for each (i, j) ; also, $B_i \cap C_j$ is disjoint from $B_k \cap C_\ell$ so long as $(i, j) \neq (k, \ell)$. Thus $B \cap C$ is a finite disjoint union of sets of \mathcal{H} , which means it belongs to \mathcal{A} .

Finally, we show (ii). As before $B = \cup_1^n B_i$ where each B_i is in \mathcal{H} and B_1, \dots, B_n are disjoint be an arbitrary element of \mathcal{A} . Then

$$B^c = \bigcap_{1 \leq i \leq n} B_i^c.$$

each B_i^c is in \mathcal{A} by property (2) of \mathcal{H} . But we just showed that \mathcal{A} is closed under finite intersections, and so B^c is in \mathcal{A} also. \diamond

5. Coin tossing space. Think of a coin with one side (heads) labelled 1, and the other side (tails) labelled 0. The set $\Omega \triangleq \{0, 1\}^\infty$ of all infinite sequences $\omega = (\omega_1, \omega_2, \dots)$ of 0's and 1's is the set of possible outcomes for the idealized experiment of tossing the coin a countably infinite number of times. For each positive integer n , let \mathcal{A}_n be the collection of all subsets B of Ω of the form: there exists a subset $C_B \subseteq \{0, 1\}^n$ such that $B = \{\omega \in \Omega; (\omega_1, \dots, \omega_n) \in C_B\}$. In words, \mathcal{A}_n is the collection of all subsets of infinite sequences which are defined by restrictions on the first n elements of the sequence. This idea is manifest in a slicker definition of \mathcal{A}_n . For each positive integer n , let $\pi_n : \Omega \rightarrow \{0, 1\}^n$ be the *projection map*, $\pi_n(\omega) = (\omega_1, \dots, \omega_n)$. Then

$$\mathcal{A}_n = \pi_n^{-1}(\mathcal{P}(\{0, 1\}^n)),$$

where $\mathcal{P}(\{0, 1\}^n)$ is the set of all subsets of $\{0, 1\}^n$.

Proposition 2 *For each positive integer n , \mathcal{A}_n is an algebra. Also, $\bigcup_{n \geq 1} \mathcal{A}_n$ is an algebra.*

Proof. That \mathcal{A}_n is an algebra is an immediate consequence of Lemma 3 and example 1. As for $\bigcup_{n \geq 1} \mathcal{A}_n$, note first that $\mathcal{A}_n \subset \mathcal{A}_{n+1}$ for each $n \geq 1$: indeed,

$$\pi_n |_{\{0, 1\}^{n+1}} \circ \pi_{n+1} = \pi_n, \text{ so}$$

$$\pi_n^{-1}(\mathcal{P}(\{0, 1\}^n)) = \pi_{n+1}^{-1}(\pi_n |_{\{0, 1\}^{n+1}}(\mathcal{P}(\{0, 1\}^n))) \subset \pi_{n+1}^{-1}(\mathcal{P}(\{0, 1\}^n)).$$

Now it is a general fact (exercise) that the union of an increasing sequence of algebras is an algebra, and this completes the proof. \diamond

3 Construction and Representation of (σ -)algebras

The following result is left as an exercise; its proof is simply an adaptation of the proof of Proposition 1 to a general situation.

Proposition 3 *Let \mathcal{E} be a non-empty collection of subsets of a set S , and suppose that \mathcal{E} satisfies*

1. $S \in \mathcal{E}$.
2. If A and B are in \mathcal{E} , so is $A \cap B$.
3. If A is in \mathcal{E} , then A^c is a finite disjoint union of elements in \mathcal{E} .

The set of all finite disjoint unions of elements of \mathcal{E} is an algebra.

This proposition gives a method for explicitly constructing an algebra starting with an arbitrary collection of subsets. Again, the proof is left as an exercise.

Proposition 4 *Let \mathcal{A}_0 be a non-empty collection of subsets of S . Take \mathcal{A}_1 to be the set of all finite intersections of the form $\bigcap_1^n B_i$ where for each i , either $B_i \in \mathcal{A}_0$ or $B_i^c \in \mathcal{A}_0$. Let \mathcal{A}_2 be the collection of all finite disjoint unions of elements of \mathcal{A}_1 . Then \mathcal{A}_2 is the smallest algebra containing \mathcal{A}_0 .*

In general, it is not easy to specify in an explicit manner each set of a σ -algebra, if the σ -algebra is not finite. Instead, one characterizes a σ -algebra by identifying a generating set. To explain, the following observation is needed: if $\{\Sigma_\lambda \mid \lambda \in \Lambda\}$ is a nonempty family of σ -algebras of a fixed set S , then the intersection $\bigcap_{\lambda \in \Lambda} \Sigma_\lambda$ is again a σ -algebra. Now let \mathcal{R} be a non-empty collection of subsets of S . The family $\Lambda_{\mathcal{R}} \triangleq \{\mathcal{C} \mid \mathcal{R} \subset \mathcal{C}, \mathcal{C} \text{ is a } \sigma\text{-algebra}\}$ is nonempty because it contains $\mathcal{P}(S)$. Define

$$\sigma(\mathcal{R}) \triangleq \bigcap_{\mathcal{C} \in \Lambda_{\mathcal{R}}} \mathcal{C}.$$

This is the smallest σ -algebra containing \mathcal{R} and is called the σ -algebra generated by \mathcal{R} .

4 More Examples

1. *Borel Sets.* We have already encountered the Borel sets in a previous lecture. If a nonempty set S is provided with a topology of open sets \mathcal{G} it is standard (but not universal) terminology to call the σ -algebra generated by the open sets the Borel sets of S . We shall use $\mathcal{B}(\mathbb{R}^d)$ to denote the Borel sets of \mathbb{R}^d with the usual topology. The open sets do not form the only generating class for $\mathcal{B}(\mathbb{R}^d)$. The next proposition lists some others.

Proposition 5 *Each of the following classes of sets generates the class of Borel sets of \mathbb{R}^d :*

- (i) *The set of closed subsets of \mathbb{R}^d ;*
- (ii) *The set of open balls of \mathbb{R}^d ;*
- (iii) *The set of open rectangles (sets of the form $(a_1, b_1) \times \cdots \times (a_d, b_d)$) of \mathbb{R}^d ;*
- (iv) *The set of bounded, right half-open rectangles of \mathbb{R}^d (see section 2, example 4).*
- (v) *All sectors, which are defined here to be sets V of the form $V = (-\infty, b_1) \times \cdots \times (-\infty, b_d)$, where $-\infty < b_i < \infty$ for each i .*

Proof. Since each closed set is, by definition, the complement of an open set, and σ -algebras are closed under complements, the σ -algebra generated by closed sets is contained in $\mathcal{B}(\mathbb{R}^d)$. By the same principle, $\mathcal{B}(\mathbb{R}^d)$ is contained in the σ -algebra generated by closed sets. This proves (i).

Items (ii) and (iii) are true because each open ball or open rectangle is a Borel set, and because each open set is a countable union of open balls and also a countable union of open rectangles.

Any right half-open rectangle can be written as an intersection of a countable number of open rectangles, so each right half-open rectangle is a Borel set. Conversely, any open rectangle is a countable union of right half-open rectangles. Thus it follows from (iii) that the right half open rectangles generate the Borel sets.

To prove (v), note first that the so-called sectors are open, so the σ -algebra they generate is contained in $\mathcal{B}(\mathbb{R}^d)$. But any right half-open rectangle $\Pi_1^d[a_i, b_i]$ is in the algebra generated by sectors. Indeed, for each j , $1 \leq j \leq d$, let

$$V_j \triangleq (-\infty, b_1) \times \cdots \times (-\infty, b_{j-1}) \times (-\infty, a_j) \times (-\infty, b_{j+1}) \times \cdots \times (-\infty, b_d).$$

Then

$$\Pi_1^d[a_i, b_i] = \Pi_1^d(-\infty, b_i) - \left(\bigcup_1^d V_j \right).$$

Thus, by (iv), $\mathcal{B}(\mathbb{R}^d)$ is contained in the σ -algebra generated by sectors. \diamond

2. *Coin tossing space, continued.* Return to example 5 of section 2. Define

$$\mathcal{A}_\infty \triangleq \sigma \left(\bigcup_1^\infty \mathcal{A}_n \right).$$

This is the smallest σ -algebra containing all subsets defined in terms only of a finite number of tosses. We showed in section 2, that $\bigcup_1^\infty \mathcal{A}_n$ is an algebra. It is *not* a σ -algebra and hence \mathcal{A}_∞ is strictly larger than $\bigcup_1^\infty \mathcal{A}_n$. Indeed, any singleton set is in \mathcal{A}_∞ but not in $\bigcup_1^\infty \mathcal{A}_n$. For example,

$$\{(1, 1, \dots)\} = \bigcap_1^\infty \{\omega; (\omega_1, \dots, \omega_n) = (1, \dots, 1)\}$$

is in \mathcal{A}_∞ but not in $\bigcup_1^\infty \mathcal{A}_n$.

Here is another set in \mathcal{A}_∞ but not in the generating algebra (exercise!). It is of interest in probability theory. Let $0 \leq p \leq 1$.

$$C \triangleq \left\{ \omega; \lim_{n \rightarrow \infty} \frac{1}{n} \sum_1^n \omega_j = p \right\}.$$

5 Monotone Class and π - λ System Theorems

One can give a construction of a σ -algebra recursively by adding at each step countable unions and complements of countable unions to the class of subsets previously constructed. In general, however, a transfinite recursion up to the first uncountable ordinal is required to complete the construction. An outline of such a construction appears in the notes to chapter 1 in Folland's book, *Real Analysis: Modern Techniques and their Applications*.

As we develop measure and integration theory, we will need to deduce properties of a σ -algebra from corresponding properties of its generating set. Fortunately, transfinite induction can be avoided at these junctures. Instead, we employ one of the following principles.

Theorem 1 (*Dynkin's π - λ system theorem.*) *Let \mathcal{E} be a non-empty collection of subsets of S closed under finite intersections. Let \mathcal{A} be a collection of subsets of S such that*

1. $S \in \mathcal{A}$.
2. If A and B are in \mathcal{A} and $A \subseteq B$, then $B - A \in \mathcal{A}$.
3. If $A_1 \subseteq A_2 \subseteq \dots$ is an increasing sequence of sets in \mathcal{A} , then $\bigcup_1^\infty A_n$ is in \mathcal{A} .

If $\mathcal{E} \subseteq \mathcal{A}$, then $\sigma(\mathcal{E}) \subseteq \mathcal{A}$.

A collection of sets \mathcal{E} closed under finite intersection is called a π -system, while a collection \mathcal{A} satisfying conditions (1), (2), (3) of Theorem 1 is called a λ -system. The theorem says that a λ system containing a π -system must also contain the σ -algebra generated by the π -system.

Here is an application that we will see later in greater generality. Fix any x in \mathbb{R}^2 . If A is a subset of \mathbb{R}^2 , let $A_x \triangleq \{y; (x, y) \in A\}$ denote its *section* at x . We want to show that if A is a Borel subset of \mathbb{R}^d then A_x is a Borel subset of \mathbb{R} . To this end define

$$\mathcal{A} \triangleq \{A \in \mathcal{B}(\mathbb{R}^2), ; A_x \in \mathcal{B}(\mathbb{R})\}.$$

Restated, we want to prove $\mathcal{A} = \mathcal{B}(\mathbb{R}^2)$. Since by definition $\mathcal{A} \subseteq \mathcal{B}(\mathbb{R}^2)$, it is enough to prove that $\mathcal{B}(\mathbb{R}^2) \subseteq \mathcal{A}$. However, we note that \mathcal{A} is certainly a λ -system. This is easy to check noting that $(A - B)_x = A_x - B_x$ and $\left(\bigcup_1^\infty A_n\right)_x = \bigcup_1^\infty (A_n)_x$. Now the collection \mathcal{H} right half-open rectangles of \mathbb{R}^2 form a π -system; see (4) of example 4 in section 2. The section of the rectangle $[a_1, b_1) \times [a_2, b_2)$ is the empty set if $x \notin [a_1, b_1)$ and otherwise is simply $[a_2, b_2)$, which is a Borel set. Hence $\mathcal{H} \subset \mathcal{A}$. Theorem 1 then implies that $\mathcal{B}(\mathbb{R}^2) = \sigma(\mathcal{H}) \subseteq \mathcal{A}$.

Here is another theorem similar to Theorem 1. A collection \mathcal{M} of subsets is called a *monotone class* if it is closed under countable increasing unions and countable decreasing intersections.

Theorem 2 (*Monotone Class Theorem*) *Let \mathcal{R} be an algebra. If \mathcal{M} is a monotone class and \mathcal{M} contains \mathcal{R} , then $\sigma(\mathcal{R}) \subseteq \mathcal{M}$.*

We will sketch the proof of the π - λ system theorem. The proof of the Monotone Class theorem is similar and is left as an exercise.

Proof of Theorem 1. Let \mathcal{G} be the *smallest* λ -system that contains \mathcal{E} , so that $\mathcal{G} \subseteq \mathcal{A}$. You can check that \mathcal{G} is well-defined, in the same way as the smallest σ -algebra containing a collection of sets is well-defined. We will show that $\mathcal{G} = \sigma(\mathcal{E})$. Since \mathcal{G} is certainly contained in $\sigma(\mathcal{E})$, because any σ -algebra is a λ -system, it suffices to prove $\sigma(\mathcal{E}) \subseteq \mathcal{G}$.

The main burden of the proof is to show that \mathcal{G} is closed under finite intersections. The first step is to show that for any $A \in \mathcal{E}$ and any $B \in \mathcal{G}$, $A \cap B \in \mathcal{G}$. To do this, fix an arbitrary A in \mathcal{E} and let

$$\mathcal{C}_A \triangleq \{B; A \cap B \in \mathcal{G}\}.$$

Since \mathcal{E} is closed under finite intersections and is contained in \mathcal{G} , \mathcal{C}_A contains \mathcal{E} . Also, \mathcal{C}_A is a λ -system, as one can easily check using that \mathcal{G} is λ -system. Hence by definition of \mathcal{G} as the minimal λ -system containing \mathcal{E} , $\mathcal{G} \subseteq \mathcal{C}_A$. Since A in \mathcal{E} was arbitrary, it follows that $A \cap B \in \mathcal{G}$ whenever $A \in \mathcal{E}$ and $B \in \mathcal{G}$.

Now take any A in \mathcal{G} and define \mathcal{C}_A as before. We have just shown that $\mathcal{E} \subseteq \mathcal{C}_A$. So, since \mathcal{C}_A is again a λ -system, $\mathcal{G} \subseteq \mathcal{C}_A$. Since A in \mathcal{E} was arbitrary, it follows that $A \cap B \in \mathcal{G}$ whenever $A \in \mathcal{G}$ and $B \in \mathcal{G}$.

The proof is finished by showing that a λ -system closed under finite intersection is a σ -algebra, for this will then imply $\sigma(\mathcal{E}) \subseteq \mathcal{G}$. But by properties (1) and (2) of a λ -system, it is closed under complements. Thus, if it is closed under finite intersections, it is also closed under finite unions and is hence an algebra. But an algebra closed under increasing countable unions is actually closed under any countable union, as the identity

$$\bigcup_1^\infty A_n = \bigcup_{n=1}^\infty \left[\bigcup_{i=1}^n A_i \right]$$

shows, and is thus a σ -algebra. \diamond .